DOCUMENT RESUME

ED 218 242 SP 020 286

AUTHOR	Troxel, Vernon; Foreman, Jane
TITLE	Mathematics in the Elementary School Curriculum: A Syllabus for a Field-Based Course.
PUB DATE	80
NOTE	111p.
EDRS PRICE	MF01/PC05 Plus Postage.
DESCRIPTORS	Diagnostic Teaching; *Elementary School Mathematics; Higher Education; Lesson Plans; Mathematical Concepts; *Mathematics Curriculum; *Mathematics Instruction; *Preservice Teacher Education; Reading Instruction; Sequential Learning; Student Evaluation; Teacher Interns; *Teaching Methods
IDENTIFIERS	Teacher Corps

ABSTRACT

This syllabus was designed as an elementary school mathematics methods course and is used with interns in a Teacher Corps project at Oklahoma State University as a part of their practicum experience. The course is conceived as a blend of directed readings, seminars, independent study, workshops, and practica. The syllabus is divided into four parts. Part One provides background information on instructional planning, strategies, and evaluation to be used in conjunction with the remaining parts of the syllabus. Part Two provides information about concepts of number, addition, and subtraction. The intern is required to observe mathematics instruction in a primary grade classroom before undertaking study of this section. Part Three is designed to provide information about multiplication, division, and fractions; the intern is required to observe instruction in an intermediate or upper grade classroom before working on this section. Part Four deals with topics from geometry and measurement that are commonly included in the elementary school mathematics curriculum. Selected readings are included on building reading skills in the mathematics class and improving mathematics verbal problem-solving ability through reading instruction. (JD)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED218242

MATHEMATICS IN THE ELEMENTARY SCHOOL CURRICULUM:
A SYLLABUS FOR A FIELD-BASED COURSE

Prepared by

Vernon Troxel
Oklahoma State University

and

Jane Foreman
Shawnee Public Schools

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

☑ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

- Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

VERNON TROXEL

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

SP 020 286

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum
(special syllabus - Teacher Corps Project)

VET
082580

THE COURSE AND THE CLASS

This is one of several courses in mathematics education at OSU. Its emphases are curricular sequences and instructional materials. Some attention is given to techniques of teaching and evaluation. The content is organized, for the most part, around the most common instructional topics in elementary school mathematics. Teaching techniques are given far greater emphases in the companion course: C&IED 5252 - Teaching Mathematics in the Elementary School.

The class is unconventional. It has been designed to be embedded in your practicum experience. It can be thought of as blend of instruction through directed readings, seminars, independent study, workshops and practica. Your work will be guided by a syllabus. Your instructor will schedule periodic group and individual meetings with you.

The syllabus is divided into four parts. Part One is designed to provide background information with broad applicability to the content of the remaining parts. They are organized around common instructional topics. Part Two is designed to provide information about concepts of number, addition, and subtraction. You are required to observe math instruction in a primary grade classroom before undertaking study of this part. Part Three is designed to provide information about multiplication, division, and fractions. You are required to observe instruction in an intermediate or upper grade classroom before undertaking study of this part. You may find that you have to skip around in the syllabus.

OBJECTIVES

The stated objectives for this course are these:

1. To identify common instructional topics in elementary school mathematics.
2. To determine prerequisites for the study of an instructional topic.
3. To establish a pupil's readiness to study an instructional topic.
4. To arrange examples in a sequence for learning.
5. To select and devise appropriate instructional models of mathematical concepts and principles.
6. To identify modes of instruction.
7. To select and devise measures of accomplishment.
8. To formulate daily and longer plans.
9. To secure supplementary materials.

10. To diagnose computational errors.

11. To select or devise records of achievement.

12. To demonstrate proficiency in the use of instructional materials.

TEXTBOOKS

Three sources will be used as basic references in this course. Others will be available in the resource center. The basic references are these:

1. Ralph T. Heimer and Cecil R. Trueblood. Strategies for Teaching Children Mathematics. (Reading, Mass.: Addison-Wesley Pub. Co., 1977)

    This text has ten chapters. Two provide background information. Eight are organized around instructional topics. Each chapter includes an overview, behavioral objectives, a theory section, a practice section, an exercise set, and an extensive bibliography.

2. Robert B. Ashlock. Error Patterns in Computation. (Columbus, Ohio: Charles E. Merrill Books, 1977)

    This book provides practice in identifying, classifying, and remedying computational errors. It is in a semi-programmed format.

3. Marilyn N. Suydam. Evaluation in the Mathematics Classroom. (Columbus, Ohio: ERIC Information Analysis Center, 1971)

    This resource guide is a direct and simple discussion of guidelines and techniques for the evaluation of instruction.

ASESSMENTS

1. Exercises. Ten exercises from each chapter in Strategies for Teaching Children have been selected for responses by you. Each set will carry a maximum of 25 points. (Units I, II, IV-XI)

2. Units. You will be asked to teach two units. One will be on an instructional topic of your choice. The other must be on problem solving. Each unit will carry a maximum of 100 points. (Units I, II)

3. Comparison of Introductory Lessons. You will be asked to teach an introductory lesson in two different ways and compare your responses. This comparison will carry a maximum of 20 points. (Unit II)

4. Model Test. You will be asked to devise a test, including an outline of specifications, that includes five types of test items. It will carry a maximum of 10 points. (Unit III)

5. Piagetian Tasks. You will be asked to administer four of these and report your findings and conclusions. Your report will carry a maximum of 20 points. (Unit I)

GRADING

Grades will be based on point-ranges. Points will be derived from meeting the requirements noted above.

A	426-500
B	351-425
C	276-350
D	201-275
F	0-200

Attendance at seminars is required. A penalty will be exacted for absences in excess of two.

CAUTION

Set a target for submitting each of the assessments. You may find it too easy to procrastinate. A score card is attached to help you keep your tasks before you.

Curriculum and Instruction 4152
Mathematics in the classroom
(Special Syllabus - Teacher Corps Project)

VET
082580

SCORE CARD

Assessment	Target Date	Sub-mitted	Score
Exercise Set - Unit I			
Commentary on Piagetian Tasks			
Commentary on Instructional Topic			
Exercise Set - Unit II			
Commentary on Problem Solving Unit			
Commentary on Introductory Lesson			
15 - Item Test			
Exercise Set - Unit IV			
Exercise Set - Unit V			
Exercise Set - Unit VI			
Exercise Set - Unit VII			
Exercise Set - Unit VIII			
Exercise Set - Unit IX			
Exercise Set - Unit X			
Exercise Set - Unit XI			

C&IED 4152

MATHEMATICS IN THE ELEMENTARY SCHOOL CURRICULUM

(Special Syllabus for interns in OSU/Shawnee Teacher Corps Project)

Prepared by

Vernon Troxel
Oklahoma State University

and

Jane Foreman
Shawnee Public Schools

Part One

This part includes three units. They are designed to provide background information with broad applicability to the content of the remaining units.

The first unit provides an overview of planning for instruction. The major items of content are a review of establishing content and developmental readiness, establishing the intent of your instruction, and ways of accommodating individual differences.

The second unit provides an introduction to instructional strategies. The major items of content are a categorization of types of lessons, broad categories of technique, reading in mathematics, and analysis of pupil errors.

The third unit provides a quick review of evaluation. The major items of content are sources of evaluative information, the uses of evaluative information, tips on developing tests, and grading.

The commentary by the authors of this syllabus is more extensive in this part than it will be in succeeding parts. Our commentary takes the place of the lectures that would be a part of the course if taken on campus.

Unit 1. Planning for Instruction

A. Materials
  1. Professional textbook
    a. Heimer and Trueblood
  2. Teacher's edition of pupil text (Secure from a cooperating teacher)
  3. Handouts (at end of unit)
    a. Piagetian tasks
    b. A Model for Individualizing Instruction

B. Study
  1. Directed observation - Thumb through the preliminary pages of the teacher's edition. Pay some special attention to the scope and sequence chart. Note that the content is divided into strands. Also note that you can determine the level at which an instructional topic is introduced and sometimes, reviewed. Most teacher's editions provide a plan for individualizing instruction. Typically, they are based on progressing through the textbook at different rates.
  2. Basic reading assignment
    a. Read the synopsis and objectives on page 2 of H & T.
    b. Read pages 3-16 of H & T.
  3. Comment
    a. Readiness for Learning. "Readiness" is one of those terms that is loaded with meaning. Readiness can be in terms of mental or physical development, prerequisite information, mental set, emotional receptivity, social preparedness, or motivational state, to mention the more prominent referents. Two of these, which are given considerable attention in elementary school mathematics (ESM), will be discussed below.

The stages in the development of intellectual structures as proposed by Jean Piaget are important to teachers of elementary school mathematics. Refer to the chart on page 3 of H & T. Note that the elementary school years of children (ages 5-11, roughly) overlap the concrete operational stage of development. Yet, since these age guides are norms, some children can be expected to be in the preoperational stage, others will reach the formal operations stage, at least in some structures.

An example may serve to give an idea of the importance of mental developmental structures. In the early days of the so-call "new math," some curriculum developers sought to introduce subtraction by drawing on its inverse relationship with addition. A readiness activity was the use of open addition sentences like $4 + \boxed{} = 7$. First grade teachers would "tear their hair"as pupils would respond $4 + \boxed{7} = 7$ or $4 + \boxed{11} = 7$. This is related to what Piaget has called "the inclusion relation." Before the age of 7, most children have difficulty in thinking of three numbers simultaneously.

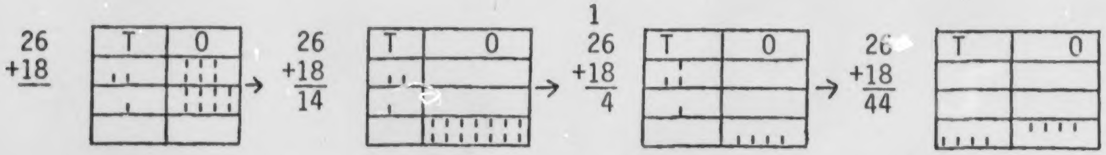
The handout, "Piagetian Tasks," was prepared by a student in one of the writers' graduate classes. She has extracted several of the more pertinent tasks. You will find her abridgement of use as you begin to determine which tasks you will administer. Richard Copeland has written two books, which are in the resource center, that provide more information. In Math Activities for Children: A Diagnostic and Developmental Approach, he describes 62 tasks, explains the levels of performance on them, and then provides some implications for teaching. In How Children Learn Mathematics: Teaching Implications of Piaget's Research, he provides greater depth to implications for teaching. We recommend that you make yourself familiar with them.

All curricula in elementary school mathematics, at least those with which we are familiar, are built around hierarchies. These hierarchies are based on the assumption that for any new learning, like the addition of two two-digit numbers with renaming, the learner has mastered all prerequisite learning. Heimer and Trueblood give emphases to these hierarchies in chapters three through ten. The hierarchies are of utmost importance to the teacher who uses any of the variations of diagnostic-prescriptive teaching.

It is possible to establish a hierachy that goes back almost to a child's conception. More practically, they are structured like the one on page 15 of H & T. For the day-to-day work of a teacher, the idea of immediate prerequisites is of even more value.

Let's take an example to illustrate the idea of immediate prerequisites. Suppose that you plan to introduce the addition of two two-digit numbers with renaming and that you will use place value pockets for the physical referent. Let's be even more specific. Suppose that the initial example is motivated by a problem that yields 26 + 18 = ☐ as the number question. The new learning here is the use of renaming ten ones as one ten and then adding three numbers in ten's place as a part of the steps in finding in the sum.

The sequence of information that is displayed as model for pupils to use would look something like this:



The pupil will have to know how to write a number question in computational (vertical) form. The addends will have to be represented on the place value pocket chart. (Note that an understanding of place value underlies both of these immediate prerequisites). The addition of the numbers in one's place (6 and 8) requires a knowledge of basic addition facts. The model, the place value pocket, will show 14 ones. The 14 ones will have to be renamed as 1 ten and 4 ones. The teacher will likely show the use of the "carry" number while modeling the renaming on the pocket chart. The addition of the numbers in ten's place (1, 2, and 1) requires competency in what is called column addition. Thus, these are immediate prerequisites: use of a place value pocket chart, computational form for addition, the concept of addition as joining, basic addition facts, renaming of ones as tens and ones, and column addition. This brief scenario is somewhat more abstract than you will find proposed in Unit V. However, it suffices for an introduction.

If a pupil does not have these prerequisites, failure (at least less than total success) is assured. Suppose that even though the pupil masters the new learnings, she has only 90 percent mastery of renaming. The addition of two two-digit numbers will theoretically be only 90 percent accurate because of deficiencies in an immediate prerequisite. Suppose that in addition to that deficiency the pupil knows only 90 percent of the basic facts, then the upper bound for success on this topic is 81 percent (90% of 90%).

How does a teacher know whether all pupils are ready to undertake a topic? You make some "best guesses" on some matters, check your records on others. A part of the art of teaching is keeping information about individual pupils in mind. Records are useful, but record keeping can become burdonsome. Tests are useful, but testing is time consuming. The sure test of readiness is the teaching of a topic. Yet, you do not want to undertake a topic without assurance of considerable success. That is why you must consider the readiness of your pupils as you plan to introduce a topic.

b. Establishing Intent. Heimer and Trueblood do a conventional job of explaining objectives and ways of categorizing them. Some comment is appropriate.

A comparision of the two taxonomies on pages 10 and 11 will reveal a parallelism in the structures. The NLSMA hierarchy has "computation" inserted between "knowledge" and "comprehension." Routine computation can be performed without much understanding, as implied by the category of "comprehension." Elements of "analysis," "synthesis," and "evaluation" are subsumed under "analysis" in the NSLMA hierarchy. Probably the reason for this is that the number of objectives at these higher levels is very limited in the elementary school curriculum.

Some students in teacher education have some difficulty with some of the descriptions. Let us discuss some of them in greater detail.

A fact establishes a one-to-one relationship. For the most part, those facts of terminology relate to definitions. Do note that emphases is on recall, not use of understanding. Before they started to school the children of one of the writers of this syllabus could state this fact: "Addition is an operation on two numbers in a set of numbers resulting in a number in the same set of numbers." Even some adults have difficulty understanding that definition. Most such definitions relate to concepts, like "any number to be added is called an addend" or "an arc is a part of a circle." Facts related to symbols are like these: " + is the sign for addition" or "the symbol for 'greater than' is >." Conventions are few. They include the My Dear

Aunt Sally rule (multiply before dividing before adding before subtracting) and "work from the inside out when removing parentheses."

Computation has one catchpoint. The definition fits constructions in geometry, which are not dealt with elsewhere in this taxonomy. For example, the bisection of an angle requires the straightforward manipulation of problem elements according to rules the subjects presumably have learned.

Comprehension involves more than straight recall of a fact and less than the use of information in an application. For example, a pupil might be able to define a rate pair (fact) and pick rate pairs out of several expressions (comprehension) but not be able to formulate a rate pair from a verbal problem. Also note that comprehension may include the translation of problem elements from one mode to another. This is involved in translating a verbal problem into a mathematical sentence. (Note: A solution is not a part of the translation.)

The major distinction between *application* and *analysis* in this hierarchy is whether the concepts are used in a routine or non-routine manner. Because of the "catchbag" nature of the analysis category, it includes problem formulation and mathematical modeling, which are much like synthesis. An example of problem formulation is when a pupil is asked to formulate a verbal problem that could be solved from a given mathematical sentence. Mathematical modeling is well illustrated in the two examples on pages 12 and 13.

We trust that you are familiar with behavioral objectives as described in H & T. Compare these with those in the teacher's edition of your pupils.text. Textbooks typically include only statements of required performance.

Take another look at Example 1 on page 9. This represents an objective in which the stimulus (the given) is at the concrete level and the response (the required performance) is at the abstract level. Theoretically, nine variations could be formulated on this objective.

Concrete, semi-concrete, and abstract refer to a continuum on which representations of mathematical ideas fall. In a rough way, one can think of concrete representations as being objects that can be manipulated, semi-concrete as being drawings or diagrams of those objects, and abstract as being words and symbols.

In Example 1, the stimulus is a set of objects (concrete) and the response is a word (abstract). One variation is this: "Given drawings of sets of nine or fewer objects, (semi-concrete) the pupil will write the numeral (abstract) for the number of objects. Eighteen (or more) out of twenty instances will suffice." Try to work out the other seven variations on this objective. Curriculum guides, like textbooks, usually give a terminal objective. Others may be important in developing a topic or in diagnosing difficulties.

c. Individual Differences. Heimer and Trueblood make only a tangential reference to accommodating individual differences among children in a class.

Children differ in about any way you can imagine. Moreover, their development differs among various traits. Fortunately, the distribution of most attributes is bell-shaped rather than rectangular or U-shaped. For that reason, most instruction is directed to the huge middle of any class. Still, differences exist. Since the beginnings of graded schools, educators have sought ways to accommodate these differences. Frankly folks, we know of no "sure fire" way of individualizing instruction.

Over the years, a variety of ways have been tried in efforts to individualize instruction. The efforts cluster around three broad categories of plans: individual instruction, small group instruction, and all-class instruction.

The most obvious form of individual instruction is tutoring. The practicality of tutors for individuals for developmental instruction in schools is obvious. (Using older pupils as tutors in particular circumstances has been proven to be beneficial to both the tutor and tutee). Moreover, most educators agree that children should have an opportunity to try out their ideas on their peers. A variation on tutoring involves the conventional textbook. Pupils sit down with a textbook and work through it from the first page to the last page. They submit their work for checking as soon as each page is completed. If they run into difficulty, they seek help from the teacher or aide. One big problem with this plan is that textbooks are not designed to be used in this manner. An inspection of your teacher's edition will reveal that the introductions to most topics are described in the prepage activities and are designed for adult guidance. Another problem is that many children need a change of pace in presentations.

Programmed instructional materials have been written to provide individual instruction. We know of only one text series that was written in this form. It did not gain wide acceptance. More commonly, programmed materials are written for special topics, for remediation, or for enrichment. One problem has been the fact that these materials are typically pencil and paper only, making no use of manipulatives. A variation on programmed textual materials are task cards. These direct pupils in the step-by-step use of manipulatives in a laboratory setting. We know of no total curriculum based on them.

Performance contracting has taken several forms. Under this plan a pupil takes a pretest that may include items dealing with both the content to be learned and the prerequisite information for its study, usually the former. The pupil is then referred to different instructional material that he studies until he thinks he has mastered the content. He then takes a posttest that has items paralleling the pertinent items in the pretest. If unsuccessful, he is cycled back through with other material. Most materials are pencil and paper, though films, filmloops, filmstrips, task cards, transparencies, and audio tapes can be used. Two large scale research and development studies have been completed that incorporated some of these features. Neither could lay claim to significantly greater achievement over conventional instruction. One resulted in lowered self-concepts.

The use of intraclass and interclass *grouping* has been tried. These groups may also be used in some combinations.

Intraclass groups may be relatively fixed over a year's time or relatively flexible. The fixed groups are commonly based on some measure of potential (like a test of academic ability) or some measure of achievement (like grades or scores on standardized achievement tests). This plan is very much like the conventional three groups used for reading instruction. One problem, as with any scheme for intraclass groups, is that the teacher must make additional preparations. Another is that some topics do not need three or four groups when they are being introduced. Still another is that a pupil may be good in one area of elementary school mathematics while poor in another. Flexible groups are designed to overcome the latter problems. New groupings are made for each topic. The record keeping and logistics are formidable. A variation on flexible groupings is the use of learning teams. Heterogeneity rather than homogeneity is the intent of a learning team. There is some evidence of success with these.

Interclass grouping has been used most commonly in the intermediate grades. Regardless of her grade level designation, a pupil studies with others who are achieving at roughly the same level. Suppose a school has two sections of each intermediate grade. A fourth grader might be in a below-third group, a grade three group, a grade four group, a grade five group, a grade six group, or an above-sixth group. Scheduling is a problem. Uneven progress through the year is another. A sixth grader studying with fourth grade pupils on third grade material can be another. The ranges of achievement on specific topics will still vary, though not as much as in a heterogeneous group.

It is possible to make some provisions for individual differences under all-class instruction. The most common way is for the teacher to make himself available to pupils as they work on daily assignments. Another is to make differentiated assignments. Still another is to encourage some pupils to work longer with concrete or semi-concrete representations while others may be using various shortcuts.

The handout, "A Model for Individualizing Instruction," is the doing of one of the writers. This model is a variation of one proposed over four decades ago as the "common fronts" approach to instruction. The writers borrowed heavily from Robert Underhill in his *Teaching Elementary School Mathematics* in developing the model. The model incorporates many of the features of the plans described above. It is applicable to all except the least capable two, three, or four pupils in most classes. We know of no research evidence that gives direct support for the plan. Refer to the handout. We shall comment on each of the boxes.

Begin by identifying the topic or topics to be learned. Ignore the subpoints. We shall deal with them in later chapters. For most of you, chapters of texbooks provide a convenient conceptualization of a topic, for example, Chapter IX in the fifth grade text of Holt School Mathematics series. The major content of Chapter IX is adding and subtracting fractions.

Follow the arrow to the box labeled "FORMULATE Objective(s)." The teacher's manual will give an abridged version of the objectives. Once you have stated (or at least have well in mind) the objectives, you are advised to create your test. Keep in mind that a test need not be of only pencil and paper items. Testing will be considered in Unit III. Also keep in mind the hierarchy of objectives.

The next box "PRE-ASSESS Learner's Information" implies a test. This is not necessarily so. You may assume some of the prerequisites. You may include a quick review of others in your introduction to the topic. You may know that none of your pupils has had the opportunity to learn about the topic. You may have checklists or performance inventories that will give the necessary information. You may need to create (or adapt) a readiness test. Many textbook series include a readiness test for most topics. Some are found in the pupil text, some in the teacher's manual, and some in supplementary booklets.

Note the branch from the line between the two boxes. If you have pupils who know what you plan to teach, they can be directed to enrichment activities.

The next box, "PROVIDE Initial Instruction," directs you to introduce the topic in a manner of your choice. For most teachers, this would be all class instruction with daily assignments. You might elect to use small groups or learning teams. Your topic might be one that lends itself to laboratory tasks.

The next step is to "EVALUATE Formatively." You can secure information from daily assignments. Though we question the wisdom of measuring success from what is intended as practice, formative information can be derived from daily assignments. Your casual observation of pupils as they respond to the introduction and as they work on assignments can provide additional information. We recommend a test that is congruent with your objectives. Many teachers use weekly tests in this manner. The tests may have as few as five to ten items. The graded tests, together with a note from the teacher, are sent home with pupils. Their parents initial the note and test before returning them to the teacher.

The next step, "DIAGNOSE," is intended to provide information for establishing as many as three groups. Those pupils who meet the criteria of mastery that are a part of your objectives can be assigned to the enrichment group. You will need to inspect the responses of pupils to the test items to diagnose further. You will be given some guidance in error analysis in Unit II of this course. Some pupils will need only practice to overcome their errors. Others just "don't get it". They will need reteaching.

When all (well, virtually all) pupils have met the mastery criterion for each (almost each) objective, you are ready to undertake the next topic. Keep in mind that even those youngsters who wind up in the re-teach group time after time deserve some of the "fun" activities of the practice and enrichment groups.

You will be asked to apply this plan as a part of your practicum. You need to keep this in mind as you assemble your idea file for the course.

C. Review

1. Turn to page 2 in H & T. Formulate a question based on each objective. These need not be in writing. Then, answer them to your satisfaction.
2. Refer to any three or four lessons in the teacher's edition. Formulate objectives in the manner recommended by H & T. Have them critiqued by a classmate.
3. Prepare any questions that you want your instructor to discuss with you.

D. Assignment

1. Immediate - respond to exercises 2,4,5,6,7,9,10,11,12,13 on pages 16-18.
2. Long range

a. Administer two of the Piagetian tasks to two children each. Try to select one child who does not possess the structure and one who does. In less that one handwritten page, report your findings.

b. Teach a unit (probably a chapter) using the plan described at the end of B-3-c above. Be sure that you have completed unit III and the appropriate unit in part Two, Three, or Four of this syllabus before teaching your unit. Submit the materials you prepared and/or used. Be sure to include your records of readiness and achievement. Write a less-than-one-page summary of your experience with this plan.

TESTS OF INTELLECTUAL MATURITY

I. Conservation of Number

Purpose:

To see if the child understands number, if he conserves.

Procedure:

Show a row of seven red blocks. Ask child to make another row with just as many blocks as the first. Then spread out the child's row and ask if there is still the same number of blocks as in the first row.

Results:

Until 6 or 7, the child thinks there are more blocks when the row is spread out. He doesn't establish a lasting equivalence-- his idea of number is still bound by preception factors.

Implications:

A child isn't ready for concentrated work with number, even counting, until he can perform the above task.

II. Additon of Classes

Purpose:

To determine when and how child understands additon as an operation.

Procedure:

Show a row of 9 beads-- 7 read and 2 blue. Ask what the beads are made of (wood). Then, what color is this (pointing at red) and this (pointing to blue). Ask, "If I made a necklace of wooden beads and a necklace of the red beads, which would be longer? Why?"

Results

Until 7 or 8, the child says there are more red beads than wooden beads. He will not change his answer even if you ask him if the red ones are made of wood and are the blue ones made of wood. He can see either the whole set or its parts but not both simultaneously. He doesn't have reversibility of thought (from whole to parts to whole again).

Implications:

Addition is only a rote process for children until they can do this task.

III. Addition and Subraction of Numbers

Purpose:

To see if the child recognizes the whole as being invariant in terms of different additive combinations such as 3 plus 3, 4 plus 2, and 5 plus 1.

Procedure:

Show a row of beans arranged as follows:

The child is asked to pretend they are pieces of candy. Today he is to have 4 in the morning and 4 in the afternoon. He can have the same number tomorrow, but since he won't be as hungry he can have one in the morning and 7 in the afternoon. At this point, remove 3 of the beans from the set on the left and put with the set on the right to represent a second day. Ask if he will have more one day or the other or the same.

Results:

Until 6 or 7 years, child thinks there is more on second day because of the big lot (7). He lacks reversibility of thought.

Implications:

The child must have reversible thought and be able to consider 3 numbers simultaneously before he can meaningfully do subtraction which Piaget regards as being with addition one reversible system.

IV. Multiplication of Numbers

Purpose:

To determine when children are able to solve problems involving multiplication of numbers.

Procedure:

Show a row of 10 blue flowers, a row of 10 vases, and a row of 10 yellow flowers. Ask the child to put a blue flower in each vase. Then remove them and ask him to put a yellow flower in each vase. Remove them and ask if there is the same number of blue flowers as yellow flowers. Then ask if we put all the flowers in the vases with the same number in each vase, how many would there be in each vase?

Results:

The child can perform this task at about 6 to 7 years.

Implications:

Multiplicative operations develop at the same time as additive operations in the mind of the child. This, despite the fact that multiplication isn't usually taught until the third or fourth grade. It would seem that we start addition too soon and multiplication too late.

IV. Concept of Fractions

Purpose:

To study the child's concepts of fractions.

Procedure:

Show a paper cutout of a cake and ask the child to draw where you would divide the cake so you could give two people the same amount. Then try it with fourths, fifths, and sixths.

Results:

Six to seven year olds and a very few four to five year olds can do this for halves and thirds. Seven to nine year olds can usually do fourths, fifths, and sixths.

Implications:

Children may be ready for fraction concepts earlier than they are usually introduced in elementary schools.

V. Beginning of Linear Measurement

Purpose:

Measuring length with a ruler is a much more complex problem for a child than most adults realize. This activity shows the stages through which children go during their first linear measuring.

Procedure:

Show a tower of different sized blocks, approximately 30 inches high. Ask the child to build another tower of equal height on another table (which is of a different height.) Place a screen between the two tables so the child can't compare towers simultaneously, but allow his to go and look at the model tower as often as he wishes.

Results:

From 6 to 7 years, the child tries to measure with his hands, putting one hand at the bottom and one at the top and walking to the other tower with his hands held apart. Between 7 and 9, he uses a stick or ruler shorter than the tower to measure with.

Implications:

This task indicates children are being started on measurement tasks long before they are ready.

VI. Conservation of Distance

Purpose:

To show that the child cannot do linear measurement until he realized that the distance between two objects remains constant when another object is placed between them.

Procedure:

Two dolls of same size are placed on a table about 2 feet apart. The child is asked if they are near or far apart. Place a cardboard screen halfway between the two dolls and ask if they are still near or far apart to see if he thinks distance has changed.

Results:

Until about seven years of age, the child thinks distance is less, that the screen has taken up some of the distance or space. He doesn't conserve distance.

Implications:

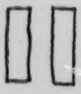
The child isn't ready for use of a ruler in any meaningful fashion until he can conserve distance.

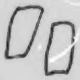
VII. Conservation and Measurement of Length

Purpose:

To show that many children think that the length of an object changes if you move it, thus making measurement by a ruler impossible.

Procedure:

Show two stick positioned as shown: 

Ask if the sticks are the same length or if one is longer. Then move one stick up or down:  and ask if they are still the same length or if one is longer.

Results:

Until about 7, most children think the length of an object changes relative to another object when it is moved. He will think that the stick that is moved up is longer because he is looking at the end farther away and thinks the stick has stretched. He doesn't conserve length.

Implications:

Children cannot measure until they can conserve length. Piaget feels that a child takes another year or so (until 8 or 9) to coordinate these different measuring abilities before he is ready to be taught linear measurement.

VIII.Conservation and Measurement of Time

Purpose:

To demonstrate that telling time requires mental development not usually found in children until 9 or 10.

Procedure:

Place two dolls of the same size side by side on a table and say they are going for a walk. When the child says go, hop the dolls along the table side by side until the child says stop. Then ask if the dolls started and stopped at the same time. Then do as before but make one doll take longer hops so that it goes farther. Ask if they started and stopped at the same time.

Results:

Below six or seven years, a child thinks the doll that went farther took more time. When asked when A stopped, was B still going and when B stopped was A still going, the child will say no. But asking again if they stopped at the same time, the child will still say no. From 6-7 to 9-10, the child will agree that the dolls started and stopped at the same time but the time duration for each doll was different - it was longer for the doll who walked farther. Not until 9 or 10, can a child answer correctly.

Implications:

Concentrated work on learning to tell time shouldn't begin until nine or ten when the child reaches the concrete operational level for understanding the idea of time. Young children can be taught the idea of time on a perceptual basis, i.e., when the little hand points to 3, it is time to go home.

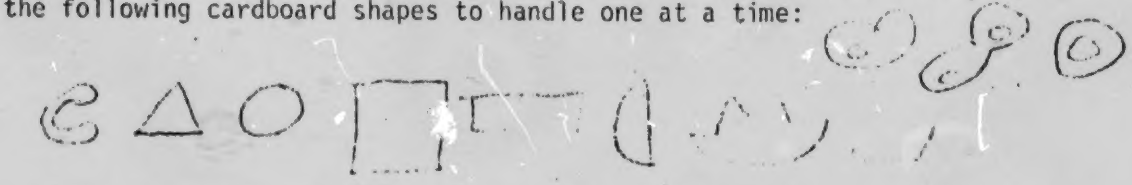
IX. Perceptual and Representational Space

Purpose:

To study the child's ability to convert sensory or perceptual impressions of an object into mental representations of the object.

Procedure:

Ask the child to put his hands through 2 holes in a screen, then give him the following cardboard shapes to handle one at a time:



Ask him to point to the object he feels on the poster.

Results:

Four to six year olds can distinguish between curved shapes and those with straight sides but no distinctions are made between squares and rectangles or circles and ellipses. At approximately six, the child can begin to differentiate although some cannot do so until 9 or 10.

Implications:

Understanding space is not just a perceptual activity. Just showing a child a shape such as a triangle doesn't insure that the child can represent mentally the idea of a triangle and understand it.

BIBLIOGRAPHY

Campbell, Sarah F., ed. A Piaget Sampler: An Introduction to Jean Piaget Through His Own Words. New York: Wiley, 1979

Copeland, Richard W. Diagnostic and Learning Activities in Mathematics for Children. New York: MacMillan Publishing Co., Inc., 1974.

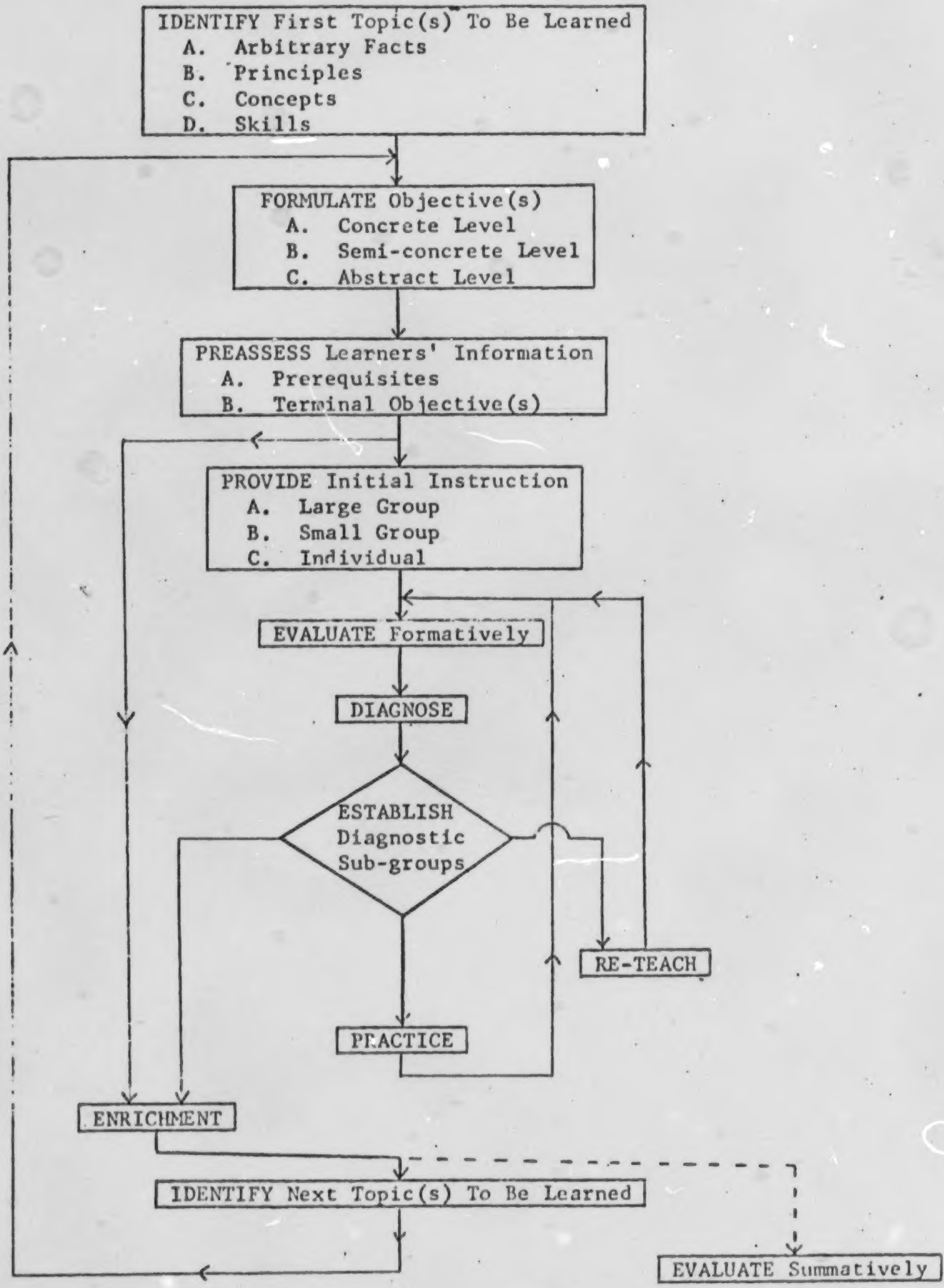
__________. How Children Learn Mathematics. London: The MacMillan Co., 1970.

Furth, Han G. Piaget for Teachers. Englewood Cliffs, N.J.: Prentice Hall, Inc., 1970.

Curriculum and Instruction 4152
Mathematics in the Elementary School

VET
082575
1,9,10

A MODEL FOR INDIVIDUALIZING INSTRUCTION


IDENTIFY First Topic(s) To Be Learned
A. Arbitrary Facts
B. Principles
C. Concepts
D. Skills
FORMULATE Objective(s)
A. Concrete Level
B. Semi-concrete Level
C. Abstract Level
PREASSESS Learners' Information
A. Prerequisites
B. Terminal Objective(s)
PROVIDE Initial Instruction
A. Large Group
B. Small Group
C. Individual
EVALUATE Formatively
DIAGNOSE
ESTABLISH Diagnostic Sub-groups
RE-TEACH
PRACTICE
ENRICHMENT
IDENTIFY Next Topic(s) To Be Learned
EVALUATE Summatively

Unit II. Instructional Strategies

A. Materials
1. Professional textbook
   a. Heimer and Trueblood
   b. Ashlock
2. Teacher's edition of pupils text (Secure from a cooperating teacher)
3. Handouts (at end of unit)
   a. Building reading skills in the mathemetics class
   b. Improving mathematical verbal problem-solving through reading instruction.

B. Study
1. Directed observation - Thumb through the pupil pages of two chapters of the teacher's edition. Concentrate on these features of the contents: (1) the nature of the content - new topic or old topic, (2) sequences - pictorial or numerical examples before the generalization or vv., (3) reading pattern - the adequacy of a left-to-right, top-to-bottom pattern, and (4) suggestions for dealing with pupil's errors.
2. Basic reading assignment
   a. Read the synopsis and objectives on page 22 of H & T.
   b. Read pages 23-42 of H & T.
3. Comment
   a. General Principles. The twelve principles stated on pages 23-26 are loaded with implication. They are well worth re-study. The principles will be referred to throughout the remainder of this course.

   b. Diagnosis/Prescription. A student teacher from one of the writers' students in this course brought this sample of errors from a survey test she had given to a third grade class. These errors are interesting in that they illustrate the fact that errors often give evidence of learning, wrong learning, but learning nevertheless. Errors of this kind are called systematic errors. A systematic error occurs at least twice. Those that occur one time are called random errors. In contrast to systematic errors, which give evidence of learning, random errors may derive from several causes: lack of learning, inattention, messiness, fatigue, or haste.

$$\begin{array}{r} 25 \\ +35 \\ \hline 69 \end{array} \quad \begin{array}{r} 34 \\ +26 \\ \hline 69 \end{array} \quad \begin{array}{r} 42 \\ +38 \\ \hline 89 \end{array} \quad \begin{array}{r} 37 \\ +33 \\ \hline 79 \end{array}$$

The sample of errors is interesting on another count. The student teacher was able to trace the source to its roots by interviewing the student and conferring with the cooperating teacher.

The test was given during the early part of the year. It had been preceded by some review. Among the topics for review was "a number has many names," for example 10 = 9 + 1 = 8 + 2 etc. Also reviewed was the addition of two two-digit numbers with renaming. It had been brought out that the only "carry number" for a sum of two addends is 1. So, the pupil put these bits of information together to rename 10 as 9 + 1, "carry" the one . . .

The sample is interesting in still another way. The lack of understanding of place value relationships is apparent. This is the source of many problems with the computational algorithms.

Now, it is not essential that a teacher know causes in order to help pupils remedy their errors. It is essential that a teacher be able to identify systematic errors. You need to become attuned to inspecting pupil work for such errors.

Read Chapter I and Appendix B in Error Patterns in Computation. You will be directed to the exercises in error analysis in units five through ten of this syllabus.

c. Reading in Mathematics. As part of your directed observation, you were asked to study your reading pattern as you read the pupil pages of a text. We assume that you could not consistently use a left-to-right, top-to-bottom pattern. The major elements for reading in the language of mathematics are discussed in the article, "Building Reading Skills in the Mathematics Class." Study it carefully.

The authors of the article provide no suggestions under "knowing what to read next." A teacher can help pupils by demonstrating reading patterns. A transparency made from a pupil page can be marked with a felt pen to show pattern(s).

d. Solving Verbal Problems. The authors of the article on reading in math classes and the authors of the professional text give some attention to the reading of verbal problems. As noted in the professional text, the matter has been studied extensively. Yet, "story problems" are a real bugaboo for many youngsters. Moreover, with the recent emphasis on "back to the basics," which has been interpreted by too many people to mean "stress computation," a decline in problem solving ability has been observed.

The 1978 survey of 71,000 youths (9-, 13-, and 17-year olds) for the National Assessment of Educational Progress supports this contention. The NAEP Newsletter (October, 1970) states,

> Most students of all three ages can answer very simple word problems . . .
>
> However, many students have difficulty with more complex problems . . .
>
> More studnets can do computation exercises than can solve problems that use the same numbers.

A systematic approach to solving the problem of solving verbal problems is described in the article, "Improving Mathematics Verbal Problem-solving Ability Through Reading Instruction." Study it carefully.

Few textbooks series provide much guidance to pupils in learning how to solve verbal problems. In some ways, this is strange in view of the importance attached to this skill. In other ways, it is understandable. A generalization from the research is that pupils learn to solve problems by solving lots of problems. It is valid, but not sufficient. Specific guidance can be given. Implied but not stated in any of these sources is the fact that a teacher is probably the best model of a problem solver in any elementary school classroom. Demonstrations of problem solving procedures by teachers are recommended practice.

The method of attack used in Henney's article is called the wanted-given approach. A sometimes used alternative is called the action-sequence approach.

The primary difference is that in the action-sequence approach a direct translation from the word sentence(s) to a mathematical sentence is made. (Remember translation as an expression of comprehension in the NLSMA hierarchy of Unit I.) The action, or implied action, is used to key the operation to be used. Using the Boy Scout problem on page 226 of Henney's article, the descriptive equation would be $5 + \square = 8$, which, obviously, is solved by subtraction. The best description of this procedure is found in Charting the Course for Arithmetic by Maurice L. Hartung, et al., (Chicago: Scott, Foresman, and Co., 1960). The research evidence for this approach is mixed. You should consider it as a possible alternative.

e. Kinds of Lessons. When you inspected the pupil's pages of the teacher's edition, you probably found some lessons designed primarily for review, some for introducing a topic, and some for practice on new learnings. Other types of lessons are those designed for reteaching and remediation. The emphasis in this, a first, course is on introducing instructional topics. However, some comment on each type is appropriate.

Review lessons are necessary. Children, like all of us humans, forget. A compounding factor to that forgetting is that children, like adults, don't have much opportunity to apply very many of their learnings in mathematics in everyday situations. (To make a point, have you had reason to divide a fraction by a fraction today? This week? Within the last month?) We do know that spaced, rather than massed review is valuable. We don't have good evidence of how many days apart the review should be nor how many items are enough.

Introductory lessons can be classified as expository of inquisitory, but more about that below. Most introductory lessons have a development part and practice part. The research evidence is overwhelming in support of prolonged attention to the developmental part of introductory lessons. Also, pupils should be actively involved in the introduction of a topic.

Practice lessons are designed to "fix" learnings from introductory lessons. An interesting activity when you have lots of time is to estimate the number of responses to practice activities that you find in a typical textbook. You will probably find a total in the 8,000-10,000 range. "Variety" is the word to keep in mind when planning practice lessons. Novel practice exercises should be a part of your idea file.

Reteaching lessons are required when a pupil just "doesn't get it" the first time. Most teachers repeat their initial development of a topic when reteaching. You may well find that these pupils have to spend more time with concrete and semi-concrete models. And this brings up an interesting point. Principle four refers to the multi-embodiments of mathematical ideas. Some pupils become confused when faced with renaming in subtraction, for example, as modeled with discs and colored rods and abacuses and t-H-T-V grids and nomographs and number lines. There is a dictum of teaching that goes, "One difficulty at a time." It applies here. Have your pupils gain some mastery with one model before using several. Of course, you capacity as a diagnostician also enters your reteaching lessons.

Remedial lessons are for special cases, where the deficiences are more pronounced and of longer duration than those requiring reteaching lessons. One important consideration is that a pupil in remediation should not be place in a situation with marked similarities to the one in which he met failure. Meeting this consideration mandates an alternative development, another algorithm, or a new model. OSU offers a course in remedial procedures: CIED 5263-Remediation in School Mathematics.

f. Strategies of Instruction. Introducing instructional topics can be viewed as being relatively simple or extremely complex. A simplification is used here. Some hint of the complicating factors will be made later.

Basically, a teacher selects her mode of instruction from four alternatives. They are shown on this grid:

SEQUENCE

MODE		Deductive	Inductive
	Expository	Gen-Auth Eg-Auth	Eg-Auth Gen-Auth
	Inquisitory	Gen-Auth Eg-Arnr	Eg-Auth Gen-Lrnr

Sequence refers to whether one or more complete examples (eg) are presented before the generalization (gen) or whether the order is reversed. The former is an inductive sequence. The latter is an deductive sequence. Mode refers to who provides the examples and generalizations. If an authority (teacher, textbook, author, script writer for a film or filmstrop) provides both, the mode is expository. If the learner(s) provide either the generalization or the example(s), the mode is inquisitory, (guided discovery). A generalization is taken here to refer to concepts, principles, rules, and procedures.

You may have noted that a third row is missing from the grid, one in which both the generalization and example(s) are provided by the learner. These situations are called pure discovery. They are rare in schools. About as close as we come to pure discovery is when we place some new instructional aids, like colored rods, before pupils and let them "play around" with them.

Now if we were to stop here and ask you to illustrate each of these modes, we would be using an inquisitory mode with a deductive sequence: A generalization (the grid) provided by an authority (us) before examples (lesson outlines) provided by the learner (you).

However we are not going to do that. Let's use some vignettes to make some comparisons. Suppose that our new learning is the concept of a number period. After suitable preliminaries, we could make a statement like this. "We mark off big numbers in sets of three digits, beginning on the right. Now, watch me mark off the number period in 4531204987". This is an expository mode using a deductive sequence. If the teacher had directed pupils to mark off some number periods without providing the example, then it would have been an inquisitory mode using a deductive sequence. For contrast, suppose that after suitable preliminaries the teacher writes four numerals having seven to ten digits each with the number periods marked off. He then says something like, "To make it easy to read big numbers like these, we begin at the right and mark off number periods of three digits. The last one to the left may have fewer than three digits." Here we have an expository mode using as inductive sequence. If the teacher had placed the examples on the board and asked pupils to figure out how he had marked them off, then it would have been an inquisitory mode using an inductive sequence.

Do note that practice exercises would likely follow any of these developmental parts of an introductory lesson. Also note that the term "number period" need not have been used at all. This is called a non-verbalized generalization.

These alternatives may seem rather pedantic. They are not. A teacher selects a mode and sequence to achieve selected outcomes. The research evidence is rather entangled. However, some rough generalizations can be made.

The expository mode and deductive sequences tend to result in greater immediate retention. that is, better test scores when the tests are given shortly after instruction. So, some form of expository instruction (like brief lectures or demonstrations, or reading the textbook) is most appropriate when immediate retention is the aim or when the learnings will be practiced in another context or reviewed with some frequency. Expository instruction commonly takes considerably less time than inquisitory instructions.

The inquisitory mode and inductive sequences tend to result in greater long-range retention and increases in learning how to learn. So, some form of discovery (inquisitory mode) is most appropriate when a learning will be practiced or reviewed but little. Few teachers at elementary school levels find use for the inquisitory mode with a deductive sequence. Learning how to learn is a worthy goal in itself. In fact, a commonly stated goal of instruction is that a learner should be able to reconstruct a learning if it is forgotten. An example might be reconstructing the procedures involved in a geometric construction or a computational algorithm.

Another factor in selecting a strategy for introducing a lesson is teacher preference. A teacher may well feel more comfortable with one mode-sequence than another. You do need to try all four as you begin to establish your teaching style.

As was mentioned above the introduction of topics can be viewed as relatively simple or extremely complex. The complexities evolve from the alternatives available for providing information and eliciting responses.

For example, information may be provided by a teacher, a text, a task card, a filmloop, a set of slides, a transparency, another pupil, a game, a puzzle, or some combination of such as these. Additionally, information can be developed from well-phrased question, and there are several levels and styles of questioning. The responses of pupils may range from full examples to almost completed examples or from full statements to "fill in one word in this statement."

C. Review

1. Refer to page 22 in H & T. Formulate a question based on each objective. Then, answer them to your satisfaction.
2. Again, thumb through a chapter in the pupils' pages of the teacher's edition. Relate what you have studied in this unit with what you find in the text.
3. Prepare any questions that you want your instructor to discuss with you.

D. Assignment

1. Immediate- Respond to exercises 1,3,5,6,8,9,10,12,18, and 21 on pages 43 and 44.
2. Long range
    a. Teach an introductory lesson of your choice to two comparable groups of pupils. With one group, use an inductive sequence.

With the other, use a deductive sequence. Write a less-than-one-page reaction to your experience.

b. Prepare a short (2-5 lessons) unit on problem solving appropriate to the level of the pupils with whom you are working in your internship . Teach it. Submit all the materials that you prepared, including lesson plans, and a less-than-one-page reaction to your experience.

Arith Tchr
Vol 21
Dec. 1974

2,8,1

Building reading skills in the mathematics class

MARY ANN HATER,
ROBERT B. KANE, and
MARY ANN BYRNE

During the past several years, the authors have jointly conducted research on reading mathematics. Mary Ann Hater is an associate professor of mathematics and education at the College of Mount Saint Joseph. She teaches mathematics and education courses and directs a mathematics laboratory for elementary teachers. Robert Kane is chairman of mathematics and science education at Purdue University. Mary Ann Byrne is assistant professor of mathematics education at the University of Georgia.

Do children know how to read mathematics books? Ordinarily not. Thus, teachers are faced with the alternatives of continually reading and interpreting instructional materials for their pupils, using such materials only as sources of teaching ideas and exercises, or teaching children to read mathematics. Since little has been done to prepare teachers to teach children to read mathematics the first two alternatives are almost always the ones that are followed.

With the increased interest in individual and small group instruction, the need for acquiring the basic skills of reading mathematics becomes more crucial. Many concepts can and should be presented effectively by means of manipulative materials, films, drawings, tapes, and other media. However, at some point a transfer needs to be made from more concrete to abstract forms. At this time the child needs to read and write in the language of mathematics and see the connection between the symbolic forms and the materials he has manipulated. In playing with marbles the child can report "four marbles added to three marbles gives seven marbles." Eventually, he must be able to respond to

$$3 + 4 = \square \qquad \text{or} \qquad \begin{array}{r} 3 \\ +4 \\ \hline \square \end{array}.$$

Children must be taught how to read mathematics; it is not a competency that they learn automatically as they are learning to read ordinary English. Most teachers have met the situation of the eight-year-old child who could not "multiply" because she could not read "$4 \times 5 = \triangle$." Her problem was not a problem with multiplication, but with the reading of the symbols. Many a sixth grader has looked at the problem $.3\overline{)28.4}$ asking, "Is it 28.4 into .3 or .3 into 28.4?" Teachers have often tended to assume that incorrect solutions to exercises and problems are due to a lack of understanding of concepts of mathematics or a deficiency in computing skills. Many incorrect solutions, however, are due to problems created because of the language of mathematics.

Reading mathematics includes the reading of words and symbols with differing relations and orders that are placed sometimes in a line, but at other times in charts, graphs, pictures, or algorithms. For example, figure 1 could appear in a young child's mathematics book. Observe that

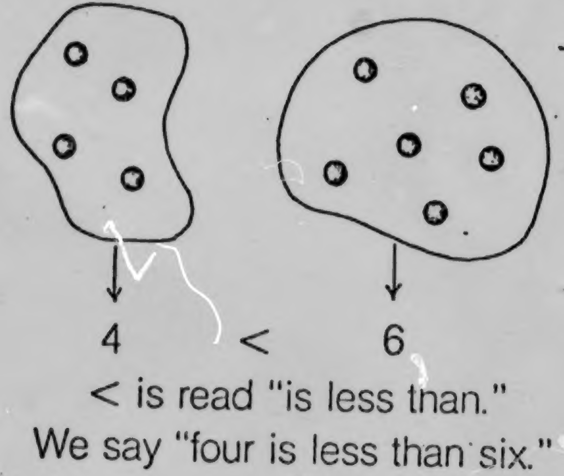
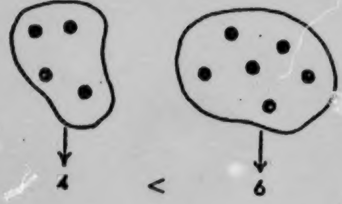


Fig. 1

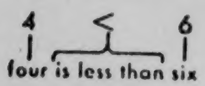
nontechnical words are used *(read, is, than, we, say),* and so are words that take on mathematical meaning *(less, four, six).* In addition, symbols that have no phoneme-grapheme connection in the usual sense must have their verbalizations memorized through usage (4, 6, <). While reading this selection the child needs to read vertically



and horizontally

$$4 < 6,$$

and compare the orderings. In addition, he needs to see the connection between what is written in symbol form and what is written in word form.



This illustration indicates some of the behaviors that a child must learn in order to read mathematics successfully. The authors have identified thirteen skills that are used in reading the language of mathematics (Kane, Byrne, Hater 1973). In the following sections these skills are introduced briefly.

Knowing what to read next

A page from a story or history book is ordinarily read from left to right. (Fig. 2.) In mathematics it is different. Symbols

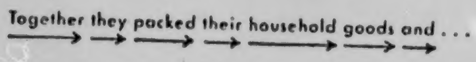
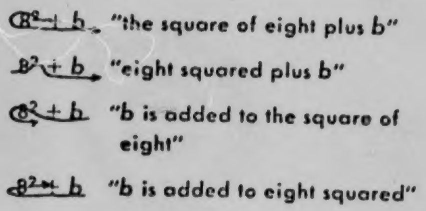


Fig. 2

are read from left to right, from right to left, from top to bottom, from bottom to top, diagonally, and using a combination of these directions. And further, a set of symbols can often be read using two or more orderings. (See fig. 3.) It takes practice in reading mathematics to know an appropriate ordering for verbalizing a set of symbols.



8^2+b "the square of eight plus b"

8^2+b "eight squared plus b"

8^2+b "b is added to the square of eight"

8^2+b "b is added to eight squared"

Fig. 3

Not only can a set of written symbols be verbalized in a variety of ways, an oral expression can be written in symbols using several different orderings. (See fig. 4.) Oftentimes the ordering in an algorithm

"What does twenty divided by five equal?"

$\frac{20}{5} = \square \qquad 20 \div 5 = \square \qquad \overset{\square}{5\overline{)20}}$

Fig. 4

is quite complex. In the algorithm in figure 5, students need to learn "where to go—next" when reading it. For example, where does a student begin to read the algorithm? Where does he read next after he reads each step? Where does it end?

$$\begin{array}{r|r} 32\overline{)738} & \\ 640 & 20 \\ \hline 98 & \\ 96 & 3 \\ \hline 2 & 23 \end{array}$$

Fig. 5

Finding the main idea

The main idea on a mathematics page may be expressed in words or symbols, or it may be left unstated for the student to discover. Whichever the case may be, there are usually clues that help the student identify the main idea. These clues are often questions, but may also be key words, pictures, symbols, examples, charts, or statements. In finding the main idea, students often need to recall previously learned principles. Sometimes these are expressed and at other times only implied. Teachers should help immature readers search for clues in determining what a page is about.

Words with many meanings

Many words that are found in mathematics books are not a part of the vocabulary of everyday speech. Other words sound like and are spelled like words from the everyday language, but take on new meanings in mathematics. The latter type is of special concern in this section. Students should be asked to respond to both the mathematical and everyday meaning of these dual-use words. (See fig. 6.) Since the word meaning from ordinary English is most often used in everyday language, it is necessary to reinforce the mathematical meaning.

I'll meet you at the *intersection* of the roads.
The *intersection* of set A and set B is empty.

What excitement to fly in a *plane!*
A *plane* has an infinite number of points.

Fig. 6

Using mathematical words

Mathematical words often denote rather complex concepts. To understand the word implies an understanding of the concept it names. Word meanings can be learned in several ways ranging from looking at examples and nonexamples of the concept to studying its definition. Regardless of the method used, an alternate method usually reinforces the meaning of the word. Students need help in using the index of their mathematics book and in sketching examples of such things as a triangle from a definition. It is important to stress the need to take time to check the meaning of mathematical words when reading before moving on.

Another important consideration is to pinpoint words that have a common stem and take on different but similar meanings with variations in prefix or suffix. The list of such sets of words is quite extensive. A few examples are shown in figure 7. The stem, prefix, and suffix might give hints as to the meaning of the word from experience gained in other contexts. *Trisector* might be comprehended by recognizing *tri* as standing for "three" in triangle and *sector* as standing for "cut" in bisector.

bisect
bisector
bisection

triangle
triangular
trisect
trisection
trisector

number
numerical
numeral

factor
factorial

Fig. 7

Seeing and understanding symbols

As a child looks at a word like *quotation*, he may not immediately be able to pronounce or understand the word. But, as he establishes the phoneme-grapheme relation he may recall the word as spoken by a teacher or classmate and come to comprehend its meaning. Such a process is not possible as a child reads symbols such as $<$, 514, 3^4, $9\overline{)72}$, 35, $\triangle ABC$, or $\overleftrightarrow{FG}$. There is no phoneme-grapheme relation between the written symbols and the spoken words for the symbols through which the child can employ word attack skills to infer the correct pronunciation. It is usually only possible to learn the connection between the written symbol and the spoken words through usage. This makes the reading of mathematics particularly difficult since the language as spoken in the classroom and the symbols for the language as written in the mathematics books are not easily translated from one to the other. Children need help with their translations. The problem is complicated by the fact that ordinarily there is not only one allowable translation, as has been mentioned earlier. A child can read $9\overline{)72}$ in several ways. Likewise, upon hearing the phrase "seventy-two divided by nine" he may write it in several ways.

Concentrated mathematical symbolism often reveals important structure. (See fig. 8.) Children will often skip over such sections out of an inability to read and derive their meanings. This is particularly true when examples of algorithms appear in context. It can be beneficial to their reading to question students as to why a certain number appears where it does, or what operation was used at a given step.

$$\begin{aligned} 397 &= 3 \text{ hundreds} + 9 \text{ tens} + 7 \text{ ones} \\ &= 3 \times 100 + 9 \times 10 + 7 \times 1 \end{aligned}$$

Fig. 8

The number of symbols added to a mathematics book that do not appear in ordinary English books is significant. Byrne (1970) has identified 153 symbols exclusive of the letters of the English alphabet that appear in books written for grades four through twelve. The way in which these symbols are related is complex. Neither the symbol nor their ways of relating with each other can be taken for granted as understood by children.

Speed of reading

Study the lines in figure 9. If you go back and read the selection aloud, you will notice that it takes longer to say the words for the passage than you may realize. Also, try writing the passage using only word forms to see how many lines you need. To read mathematics it is most often necessary to read carefully and slowly in order to comprehend the meaning of the compact symbols and technical words. Students need to learn to use different speeds as determined by the purpose of reading. As an example, when reading a section involving many symbols, a student may be told to—

Complete these: $275 + 53 = ⬡$

$53 + 275 = ▽$

By the commutative property of addition,

$$275 + 53 = 53 + 275$$

Fig. 9

(*a*) skim quickly to find out what the symbols are about,

(*b*) reread the symbols slowly to yourself,

(*c*) answer any questions and fill in the blanks.

Some factors affecting optimum reading speed include the amount and concentration of symbols, ordering of symbols, number of charts and graphs present in the lesson, number of mathematical words, and the writing style employed.

Reading with paper and pencil

When reading a history book it is often useful to take notes for future reference.

Note taking in mathematics fulfills a more vital function for it becomes a necessary ingredient in being able to move from step to step. A student may use a pencil to fill in a placeholder in an algorithm, to answer a question, to make a drawing that illustrates a definition or principle or to trace a figure so that it can be folded.

Students should be encouraged to have paper and pencil always available when reading a mathematics book so that they can make the type of notes necessary to comprehend the lesson.

Reading charts and graphs

There are several aspects of chart and graph reading that need to be considered before saying that a graph or chart is understood. Students should (1) look at the title to determine the nature of the graph, (2) find the main categories used to describe the information contained therein, and (3) identify the details of the chart or graph as they relate to the main categories. As students become skilled in the use of charts and graphs, they should be helped to make inferences from the data, realizing the tentativeness of their conclusions. It should be pointed out that both vertical and horizontal reading are needed for much of the information on charts and graphs. Students learn how to read and interpret charts and graphs by constructing them using data that they have collected.

Seeing what's on the page

There is hardly a book used that exhibits as much stereotypic page structure as a mathematics textbook. Many mathematics books begin each section only at the top of a page, use bold-face type for new words, put important definitions in color or boxes, end a page with a set of exercises that are neatly labeled. But, this is not all. As you move down a page, similar problems are indented and spaced the same.

Lines of print may even separate sets of symbols, yet the symbols are aligned. (See fig. 10.) Teachers should be very conscious of this valuable aid and help students to see that much of the structure of mathematics becomes clearer by studying the page structure of the mathematics book. Children should also be helped to see the structure of each chapter and of the book as a whole. Where are the self-study pages? the review pages? chapter tests and laboratory experiments? Is there a dictionary of terms? a table of contents? an index? Are there answers or page references to examine?

$$\begin{aligned} 436 &= 400 + 30 + 6 \\ &= (4 \times 100) + (3 \times 10) + (6 \times 1) \end{aligned}$$

Recall that we can write

$$100 = 10^2,\ 10 = 10^1, \text{ and } 1 = 10^0, \text{ then,}$$

$$436 = (4 \times 10^2) + (3 \times 10^1) + (6 \times 10^0)$$

Fig. 10

Problem solving

Reading and interpreting verbal problems in mathematics has often been equated with the larger problem of reading mathematics. It has often been said that children cannot do verbal problems because they cannot read (referring to reading ordinary English). There is a degree of truth in that statement. It is certainly necessary that students be able to read in order to solve problems, but it is not sufficient. The situation is more complex than that and needs more discussion than can be included in this article. Reading is part of the difficulty and teachers can help students develop the ability to solve verbal problems by involving them in lessons that include some of the factors necessary to successful reading and interpretation of problems. Such activities include discovering the question, identifying relevant information, translating the information from verbal to symbolic form, using concrete situations to act out problems, recognizing concepts and operations from word forms, and constructing an appropriate mathematical model (usually a number sentence, sometimes a diagram).

Helps in computing

Pictures, words, charts, and structures may be used to help children read and interpret algorithms. For example, a child may be helped with 6 × 3 by observing six sets of three or seeing the words "six sets of three each." A multiplication chart may be present in which he observes how the entries form a structure that helps him to see 6 × 3 as related to 6 × 1 and 6 × 2. As computations become more complex, algorithms help to organize the work. In a problem like the one in figure 11, students need help in knowing whether each numeral stands for a sum, difference, product, or quotient. Cycles of computation should be studied. That is, the operation of multiplication, to obtain 400, is followed by subtracting, to obtain 121, and so on.

```
4/521 |
  400 | 100
  ----|
  121 |
  100 |  25
  ----|
   21 |
   20 |   5
  ----|-----
    1 | 130
```

Fig. 11

Children need help in knowing how to read algorithms as they appear in mathematics books, so that they do not just pass over illustrative examples but use them to enhance learning.

Objects,, idea, word, symbol

Stress must be placed on the need in reading to connect a word or symbol with objects or models of the physical world in order to form ideas and derive meaning from the reading. Today there is stress on the use of concrete materials to aid in the understanding of mathematical concepts. As a student matures in his knowledge of a concept, he need not necessarily constantly refer to the models or objects themselves, but may deal with the words and symbols for the ideas. As concepts are deepened or related with other concepts it may be helpful to return to a model for a discovery.

For example, a child may be able to use a model like the one in figure 12 to understand addition of two fractions. Not only is it important for him to use the model and verbalize the rule, but also to recognize the symbols and written words as standing for the same operation.

$$\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$$

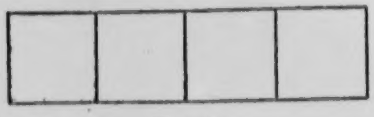


Fig. 12

Searching for meaning

While reading, children often encounter new words, expressions, and symbols that are unfamiliar to them. Children could respond by looking them up in the dictionary or glossary of terms, asking someone for help, giving up, or using context. The last alternative, that of using context, can be one of the most beneficial alternatives.

In the example in figure 13, words and symbols have been deleted. Try to respond to the deletions by using the context that remains. Note that some responses can be made because of the structure of the passage and others because of the mathematical content. Exercises such as this may

In base ________ we group by tens.

________ base six we group by sixes.

How do we ________ in base five?

1___ = (2 × 5___ + 3 = 2___ five

21 = ___ 4 × 5) ___ 1 = 41 __________

See if you can ________. 17 in

base ________.

Fig. 13

2,8,7

be incorporated in the mathematics program to help children learn to respond to a question of meaning by searching a page for similar structure or appropriate hints from context.

Using skills together

The skills that are needed to read mathematics with success have been discussed individually, but you must have noted that none of these skills stands in isolation; each is very dependent on others. Number and concentration of symbols affects speed, ordering affects the ability to read charts, using the correct meaning of a word affects the ability to find the main idea, and so on. Students should practice these skills in combination as well as one at a time.

Learning to read in the language of mathematics is a crucial competency needed by a large majority of the students in our schools. Yet the usual reading and mathematics programs do not, in general, provide the sorts of activities necessary to acquire this competency. The proper place to learn how to read mathematics is in the mathematics class. It is in the mathematics class that students naturally combine the skills of learning to read with the actual practice of reading mathematics.

References

Byrne, M. A. "The Development of a Measure of the Familiarity of Mathematical Terms and Symbols." Ph.D. dissertation, Purdue University, 1970.

Kane, R. B., M. A. Byrne, and M. A. Hater. *Helping Children Read Mathematics.* New York: American Book Company, 1973.

Improving mathematics verbal problem-solving ability through reading instruction

MARIBETH HENNEY

Maribeth Henney is an assistant professor in the elementary education department at Iowa State University, Ames, Iowa. Her responsibilities include directing a course in methods of teaching reading, and supervision of student teachers.

One area that elementary school children find difficult is solving verbal problems in mathematics. Even though they may be able to do the computational work, children find solving verbal problems most arduous. Comments such as the following are offered in evidence: from children—"Ugh! Another page of story problems!"; from teachers—"My children just seem to hate story problems. They can't read them well enough to figure out what arithmetic needs to be done."

Researchers have listed various factors that influence success in verbal problem solving. (For information concerning these factors, the reader may want to consult the sources listed at the end of this article.) Although there is lack of agreement on which factors have the greatest influence, there is consensus that children must be able to recognize words and comprehend thought units in the problem, logically interpret the problem situation, and organize the information in such a way as to lead to an answer to the stated question. Computational skill is also necessary, but it is useless without the ability to read the problem accurately.

Reading verbal problems

In mathematics, the style of writing is quite different from that usually found in narrative materials. Only short units of material are read at one time. There may be little continuity between problems. The content of verbal problems may be only a means to an end—to interest the child in working the problem.

Mathematics writing is compact; content is limited to essential ideas. The lack of rich context makes word identification difficult. Some of the words may already be familiar to children but have unfamiliar meanings in the mathematical context. Others are easily identified as belonging only to the field of mathematics and are seldom met in other reading. Such technical vocabulary must be recognized and the concepts understood if children are to understand what they read in mathematics.

A mixture of graphic and numerical symbols may break the reader's train of thought and prevent him from reading in complete thought units. The reader needs as good an understanding of the numerals, signs, and abbreviations used in mathematical writing as he would have if these concepts were all expressed in the uneconomical form of words. The reader tends to

---

For information concerning the application of the method of instruction presented in this article, the following source by the author may be consulted: "The Relative Impact of Mathematics Reading Skills Instruction and Supervised Study upon Fourth Graders' Ability to Solve Verbal Problems in Mathematics" (Ph.D. diss., Kent State University, 1968).

concentrate on the numerals in a problem, since he knows they will be needed for its solution. Different patterns of eye fixation are required for reading words and for reading numerals, the adjustment causing difficulty for some children.

Reading verbal problems requires more intense concentration than that needed for reading narrative materials. Time must be taken periodically to reflect on what has been read, to search for relations in the information given, and to apply appropriate mathematical skills for the solution of the problem. Analytical reading is necessary for recognizing limitations of the data given, determining relevancy of information, and seeing relations. The reader must then be able to synthesize or integrate his ideas and infer the appropriate operation to be used in solving the problem. This type of reading is necessarily slower than that used for reading narrative materials. Students need to learn to adapt their rate of reading to the type of material being read and to the purpose they have in mind in reading it.

Rereading is often unnecessary with narrative materials, but it is essential for successful reading of verbal problems. A quick skimming of the problem will enable the reader to get a general impression of the problematic situation and to determine the main idea. He should be able to recognize what is required in the problem, that is, what the question is. This must be kept in mind as he rereads the problem to pick out the relevant details, to see the relations between the main and subordinate ideas, to organize the data, and to infer a process for solving the problem. After the computation is completed, another rereading is necessary to check the reasonableness of the answer. Each rereading is for a different purpose and requires a different reading skill.

Developing ability to read verbal problems

Although children work with "story problems" from kindergarten on, the reading of problems may not become a stumbling block for children until about third or fourth grade. Prior to this time, problems are usually given orally or written using familiar vocabulary, simple sentence structure, and no extraneous information. The task confronting the problem solver in such a case is one of interpreting the action of the problem to determine the appropriate operation. Concrete objects or pictures are often used to help him read and solve the problem. The teacher may guide the analysis with well-timed questions and oral practice, having the child think through the solving of the problem out loud.

Verbal problems become more complicated, however, at approximately the fourth-grade level. The vocabulary used may have more than one meaning, requiring the child to choose an appropriate meaning according to the context. Sentences involve more difficult grammatical structure. Material that has no bearing on the solution is sometimes included in the problem. To solve problems the child must be able to read (recognize, comprehend) the words in the problem, visualize the situation to decide what the problem is about in general terms, recognize the question asked, note the important facts given in the problem, infer the relation among these facts which will imply the correct operation to be used in the solution, and interpret the solution obtained in terms of the question asked.

It is assumed by many authors of mathematics textbooks that since children in fourth grade or above can read fairly well, they can do more mathematical work in a verbal setting. Intermediate-grade children are usually confronted with a greater number of verbal problems than primary children are. It seems that fourth grade would be the optimum level at which instruction in reading skills should be begun. Also, intermediate-grade children ordinarily receive instruction during their developmental reading program in study skills such as finding the main idea, noting details, recognizing relevant and irrelevant information, and inferring relationships. If reading-skills instruction is given during mathematics instruction, certain reading skills learned during the developmental reading period will be reinforced and learning will more likely be seen as functional.

In order to help children realize that each subject-matter area differs somewhat in the type of reading that is required, a lesson could be devoted to examining one page from each of the types of books that are used during the day and discussing the way each page would be read, pointing out such differences as purpose for reading, rate of reading, and use of the information read. This same goal could be achieved by calling attention to the application of various reading skills as work is done during the day in each subject area.

After children become aware that reading is an important part of all subjects, they may be more receptive to reading instruction during mathematics period. The first lesson might involve only the first step in solving verbal problems—rapid reading of problems to locate the main idea of the problem situation. The child should be encouraged to state the main idea in his own words. It may be necessary for him to use pictures, diagrams, or concrete objects to clarify the meaning of the problem situation and to visualize what is happening. Children who have difficulty with this step may be allowed to dramatize the situation or make up verbal problems based on their own experiences in order to see the relationship between the verbal statements and actual occurrences.

It is suggested that problems without numbers be used for the initial instruction in reading, since children tend to give attention primarily to the numbers in a problem, trying to guess what to do with them, rather than reading the problem carefully to determine the important information and the relationships between the facts. A worksheet could be used to assist the children in recording their thoughts about the problem. Table 1 shows the format for the worksheet and the headings that could be used.

The verbal problem given in column (1) of table 1 has no numerical answer, but the children could easily visualize the situation. A more careful reading of the problem would allow the next two steps in solving the problem to follow the initial visualization. A child's reasoning might be:

> I can see all the pups. Some are black, and some are black-and-white spotted. If I take all the black-and-white spotted ones and put them in a separate pen, I can count the black ones that are left.

The steps completed so far would be recorded under the headings "Main idea," "Question," and "Important facts." In order to arrive at an answer to the question, the relation between the important facts and the question must be recognized. Children need to grasp the structure of the problem before they look for an answer. The "Relation sentence" section of the worksheet could then be completed.

It may seem that this would be asking children to do too much writing to express with words what could more efficiently be expressed with numerals and operation symbols. This procedure should be used, nevertheless, until it is certain that children have developed sufficient skill in visualizing the situation, in recognizing the information given in the problem, in determining what information they are asked to find, and in seeing the relationships that exist. A premature use of numerals and symbols may distract attention from the kinds of information to be dealt with in the problem and the relationships involved.

After some work with this procedure has been done, variables may be introduced. It is important even yet, however, that the relation sentence be read by using the phrases represented by the symbols and not just by naming the symbols. "Triangle minus rectangle equals square," for example, is not in fact a true sentence. It is true only when the phrases for which the symbols stand are read as parts of the statement. The symbols are used for econ-

Table 1
Completed steps of problem analysis

	Type of problem		
	No numbers (*1*)	*One-step* (*2*)	*Two-step unrelated* (*3*)
Verbal problem	Your dog has just had pups. Some of them are black, and some are black-and-white spotted. How many are black?	Eight Boy Scouts are going on a camping trip. One car can hold 5 Boy Scouts. How many will need to ride in another car?	You are repairing your bicycle. A new handlebar costs \$1.50, a new seat \$2.75, and a new tire \$3.85. What is the total cost of your purchases? How much cheaper is the handle bar than the tire?
Main idea	A dog had some pups.	Boys will ride in cars to go on a trip.	I'm repairing my bike.
Question	How many pups are black?	$\square$ = the number of boys who will not be able to ride in the first car	$\square$ = total cost of purchases $\diamond$ = difference between cost of handlebar and tire
Important facts	The number of pups. The number of black-and-white spotted pups.	$\triangle$ = the number of boys who will ride in the cars ∇ = the number of boys who will ride in the first car	$\triangle$ = cost of handlebar ∇ = cost of seat $\bigcirc$ = cost of tire
Relation sentence	The total number of pups minus the number of black-and-white spotted pups equals the number of black pups.	$\triangle - \nabla = \square$	$\triangle + \nabla + \bigcirc = \square$ $\bigcirc - \triangle = \diamond$
Mathematical sentence		$8 - 5 = \square$	$150 + 275 + 385 = \square$ $385 - 150 = \diamond$
Computation		$8 - 5 = 3$	$150 + 275 + 385 = 810$ $385 - 150 = 235$
Answer sentence		Three boys will have to ride in another car.	All my purchases will cost \$8.10. The handlebar is \$2.35 cheaper than the tire.

omy in writing only and do not alter the meaning of the relation sentence. It must be emphasized that these symbols can lose meaning unless continuously related to the word phrases from the problem. Sufficient experience with using only words in the relation sentence must be given before proceeding to the use of symbols in order to insure that the meaning of the symbols is not lost.

Once this process of analytically reading a problem is learned, problems with numbers may be used. The worksheet for the problem given in column (2) of table 1, would be completed exactly as was done when the problem had no numbers. There is no need for any use of numbers in the first four steps of thinking through the problem. However, in order to go beyond the relation sentence, that is, in order to find a numerical solution to the question, numbers must be used in a mathematical sentence. Each symbol of the relation sentence is read with words (according to the "Question" and "Important facts" sections) and this information is then located in the verbal problem—for example: "$\triangle$ = the number of boys who will ride in the

Table 1—continued
Completed steps of problem analysis

Type of problem		
Two-step related (4)	*Extra information* (5)	*Missing information* (6)
At your last swimming lesson, you spent time building endurance for long-distance swimming. On your first try, you were able to swim 20′, the second time 25′, and the third time 25′. What was your average swimming distance?	Your Girl Scout leader is purchasing merit badges for some of the girls in your troop. These 5 girls have earned a total of 12 badges. Each badge costs 35¢. How much will your leader pay for these badges (not including tax)?	You are providing the refreshments for your troop meeting. Candy bars are 10¢. You need one for each of 9 people. Each person will also have a can of pop. How much will the refreshments cost you?
I'm trying to swim long distances.	A leader is buying badges for her troop.	I'm buying refreshments for my troop meeting.
$\square$ = average distance I swam	$\square$ = the cost of the badges	$\square$ = the cost of the refreshments
$\triangle$ = distance I swam in first try ∇ = distance I swam in second try $\boxed{}$ = distance I swam in third try $\diamond$ = total distance I swam $\bigcirc$ = number of tries	$\triangle$ = the number of badges bought ∇ = the cost of each badge	$\triangle$ = number of candy bars needed ∇ = cost of each candy bar $\boxed{}$ = number of cans of pop needed $\diamond$ = cost of each can of pop
$\triangle + \nabla + \boxed{} = \diamond$ $\diamond \div \bigcirc = \square$	$\triangle \times \nabla = \square$	$(\triangle \times \nabla) + (\boxed{} \times \diamond) = \square$
$20 + 25 + 25 = \diamond$ $\diamond \div 3 = \square$	$12 \times 35 = \square$	$(9 \times 10) + (9 \times \text{missing}) = \square$
$20 + 25 + 25 = 70$ $70 \div 3 = 23.3$	$12 \times 35 = 420$	
My average swimming distance was 23.3′.	The leader will pay $4.20 for the badges.	

cars = 8." The appropriate numeral related to each important fact is given in the mathematical sentence.

No numeral is yet available to replace the symbol for the question. Computation indicated by the mathematical sentence must be completed. Before writing the answer sentence, another quick rereading of the problem may be necessary to determine exactly the question the problem asked. The interpretation of the numerical solution to the computation is then expressed in a brief sentence.

Multistep problems involving two or more questions, either explicitly or implicitly stated, may be solved using this same procedure. In the problem in column (3), table 1, both questions are explicitly stated. The answer to one is not dependent on the answer to the other. It does not matter which is solved first. In completing the worksheet, both questions are written in the "Question" section, with different symbols representing each one. All important facts needed for answering either question are listed, but only those needed for the question being answered are used in the relation sentence and mathematical sentence for that question. Two relation, two mathematical, and two answer sentences are written, since a separate analysis is completed for each question.

In the problem in column (4), table 1, two questions are involved but only one is explicitly stated. Only one question should be listed on the worksheet. When listing the important facts, care must be taken to assure that all important information needed to answer the question is noted. Even those facts not yet seen to be important are listed, for they will be needed in the relation and mathematical sentences.

The first relation sentence includes those important facts needed to answer the hidden question (How far did I swim altogether?). This answer then becomes important information to be used in the next relation sentence, formed in answer to the stated question in the problem. Each of the two mathematical sentences is completed in order, replacing the symbols representing the phrases in the relation sentence with numerals from the verbal problem. The computation indicated by the first mathematical sentence (answering the hidden question) must be completed in order to obtain that numerical information needed in the second mathematical sentence. Following the computation for the second mathematical sentence, the answer sentence may be written in answer to the stated question.

Children need experience in solving problems containing extra information or insufficient information as well as problems with only relevant information. The problem in column (5), table 1, contains numerical information that will not be used in finding the answer. The reading and analysis of such a problem can proceed as explained previously. However, special care should be taken to select only that information especially pertinent to the question in the problem when listing the important facts on the worksheet. Any information that is not essential for answering the question is not listed. After analysis of the problem is completed, children should be asked to specify what information, if any, was given which they did not use. This practice will help clarify the presence of "extra" information.

Analysis of the problem in column (6), table 1, would proceed as usual up to the mathematical sentence. When searching in the problem for a numeral to use for the cost of a can of pop (one of the previously listed important facts), children will be unable to find any such numerical information. The word *missing* is used in place of the numeral needed to replace the symbol for that fact. Computation cannot be completed for this mathematical sentence due to insufficient information. Therefore the problem cannot be solved.

The steps involved in the analytical reading of verbal problems should be introduced gradually, with practice to reinforce each type of thinking skill before more difficult aspects are introduced. After children have had experience with all types of problems, they should not be forewarned about the characteristics of the problems to be solved. If problems of various types are presented in each lesson, the children will be required to read critically to pick out relevant information, to avoid distraction by irrelevant information, and to recognize when limited information makes the solution impossible. Even an occasional problem in which the answer is explicitly given will challenge children to read critically.

It should by now be obvious to the reader that a large part of verbal problem solving is a reading task. If children are to become efficient in the type of reading required for solving verbal problems, special attention must be given to the instruction in and development of such reading skills.

References

Berglund-Gray, Gunborg, and Robert V. Young. "The Effect of Process Sequence on the Interpretation of Two-Step Problems in Arithmetic." *Journal of Educational Research* 34 (September 1940):21–29.

Brueckner, Leo J. "Improving Pupils' Ability to Solve Problems." *Journal of the National Education Association* 21 (June 1932):175–76.

Brueckner, Leo J., and Guy L. Bond. *The Diagnosis and Treatment of Learning Difficulties.* New York: Appleton-Century-Crofts, 1955.

Corle, C. G. *Teaching Mathematics in the Elementary School.* New York: Ronald Press, 1964.

Kramer, Klaas. *The Teaching of Elementary School Mathematics.* Boston: Allyn & Bacon, 1966.

Riedesel, C. Alan. *Guiding Discovery in Elementary School Mathematics.* New York: Appleton-Century-Crofts, 1967.

Books received

Beginner's Book of Geometry. Grace Chisholm Young and W. H. Young. Bronx, N.Y.: Chelsea Publishing Co., 1970. Pp. xvi + 222, $4.50.

A reprinting of a classic in mathematics education, written for those who guide children's learning. Gives practical methods of presenting primary geometrical ideas and theorems.

The Easy Book of Fractions. David C. Whitney. New York: Franklin Watts, 1970. Pp. 72, $3.50.

Pictures and color are used to show how everyone uses fractions.

Flowcharting. James D. McQuigg and Alta M. Harness. Boston: Houghton Mifflin Co., 1970. Pp. iii + 43, $0.75 net.

A workbook planned to help students learn how to set up a problem in a form that would make possible its solution by a computer.

Geometry: A Modern Course. Max Beberman, J. Richard Dennis, Martin S. Wolfe, and Russell E. Zwoyer. Lexington, Mass.: D. C. Heath & Co., 1971. Pp. 566, $7.80.

A text intended for the average student; developed from *High School Mathematics, Course 2* by Beberman and Vaughan. Emphasizes both discovery learning and deductive reasoning. Teacher's edition available.

The Guide to Simulation Games for Education and Training. David W. Zuckerman and Robert E. Horn. Cambridge, Mass.: Information Resources, 1970. Pp. 334.

Planned as a simple and useful guide for the potential game user. Includes an eighteen-page section on mathematical skill-development games. Another section describes simulation games in education.

Improving the Teaching of Modern Mathematics in the Elementary School, rev. ed. Willard O. Stibal. Dubuque, Iowa: Kendall/Hunt Publishing Co., 1970. Pp. v + 176.

A study guide with references, teaching helps, and a written assignment for each of fifteen topics. Major emphasis is on developing perception through mathematical structure.

Mathematics in the Making, vols. 9–12. Vol. 9, *All Sorts of Numbers,* Stuart E. Bell; vol. 10, *Sets and Relations,* Stuart E. Bell; vol. 11, *Graphs,* G. Long and K. A. Hides; vol. 12, *Statistics,* Irene Campbell. Boston: Houghton Mifflin Co., 1970. Pp. 32 each, $0.99 net each.

Booklets, some of them prepared in collaboration with the School Mathematics Project in England, providing activities for the various topics. Color and variation in page format used to add interest.

Puzzles and Teasers. Nicholas E. Scripture. New York: Van Nostrand Reinhold Co., 1970. Pp. 74, $4.95.

A variety of puzzles—mathematical, logical, crossword, vocabulary, general knowledge, oddities. Answers included.

Rara Arithmetica. David Eugene Smith. Bronx, N.Y.: Chelsea Publishing Co., 1970. Pp. xviii + 725, $15.00.

This fourth edition of *Rara Arithmetica,* first published in 1908, includes DeMorgan's *Arithmetical Books.* A descriptive catalog of early works on arithmetic; many reproduced pages.

Unit III. Evaluation

A. Materials
  1. Monograph
    a. Suydam
  2. Teacher's edition of pupils text (Secure from a cooperating teacher.)

B. Study
  1. Directed observation- Thumb through the pages of the teacher's edition of the pupils text. Observe the kinds of tests that are included: readiness, formative, summative, diagnostic. Observe the number items on each kind of test. Observe the kinds of test items that are used: short answer, multiple-choice, true-false (or other alternate choice), multiple choice, matching, essay. Observe whether any testing situations at the concrete level (either stimulus or response) are recommended.
  2. Basic reading assignments and comment- The reading assignments for this unit will be different from those found in any other units. Some comment will be made about each part of the monograph. Questions will be asked to establish purposes for your reading. Then, additional comments will be made.

    a. Introduction
      1) Comment I: We debated whether to place this unit in Part One or Part Four. We concluded that Part One is the proper positioning, primarily because we think that you should consider terminal behaviors as you plan your instruction. We also assumed that you know something about measurement and evaluation. Thus, we are approaching this unit as review, review with special application to elementary school mathematics.
      2) Questions:
        a) How does measurement differ from evaluation?
        b) What are sources of evaluative information?
      3) Read pages v.- 5.
      4) Comment II: The distinction between formative and summative evaluation is evident in the classroom vignette found on these pages. Some information is used to assess accomplishments against goals. Evaluation requires that a judgment be made. You as an educator have to make these judgments. The information that you secure provides the basis for making these judgments.
    b. Scope of objectives
      1) Comment I: The information included in this part of the reading assignment is similar to that found in Heimer and Trueblood. Of the three ways of classifying objectives, two will be familiar to you.
      2) Questions
        a) Why should a teacher have some way of classifying objectives?

tests are given shortly after instruction. So, some form of expository instruction (like brief lectures or demonstrations, or reading the textbook) is most appropriate when immediate retention is the aim or when the learnings will be practiced in another context or reviewed with some frequency. Expository instruction commonly takes considerably less time than inquisitory instructions.

The inquisitory mode and inductive sequences tend to result in greater long-range retention and increases in learning how to learn. So, some form of discovery (inquisitory mode) is most appropriate when a learning will be practiced or reviewed but little. Few teachers at elementary school levels find use for the inquisitory mode with a deductive sequence. Learning how to learn is a worthy goal in itself. In fact, a commonly stated goal of instruction is that a learner should be able to reconstruct a learning if it is forgotten. An example might be reconstructing the procedures involved in a geometric construction or a computational algorithm.

Another factor in selecting a strategy for introducing a lesson is teacher preference. A teacher may well feel more comfortable with one mode-sequence than another. You do need to try all four as you begin to establish your teaching style.

As was mentioned above the introduction of topics can be viewed as relatively simple or extremely complex. The complexities evolve from the alternative available for providing information and eliciting responses.

For example, information may be provided by a teacher, a text, a task card, a filmloop, a set of slides, a transparency, another pupil, a game, a puzzle, or some combination of such as these. Additionally, information can be developed from well-phrased question, and there are several levels and styles of questioning. The responses of pupils may range from full examples to almost completed examples or from full statements to "full in one word in this statement."

C. Review

1. Refer to page 22 in H & T. Formulate a question based on each objective. Then, answer them to your satisfaction.
2. Again, thumb through a chapter in the pupils' pages of the teacher's edition. Relate what you have studied in this unit with what you find in the text.
3. Prepare any questions that you want your instructor to discuss with you.

D. Assignment

1. Immediate- respond to exercises 1,3,5,6,8,9,10,12,18, and 21 on pages 43 and 44.
2. Long range
   a. Teach an introductory lesson of your choice to two comparable groups of pupils. With one group, use an inductive sequence.

3) Read pages 5-8

4) Comment II: Currently, the NLSMA hierarchy is used more often than the others. A fifth category, facts or factual information, is added to the bottom of the hierarchy. We encourage you to use these five categories in developing your tests.

c. Evaluation Purposes

1) Comment I: The purposes are stated very succinctly in this section. Tha main point is that one evaluates for more than one reason, assigning grades.

2) Questions

a) Why should you evaluate?

b) Why are comparisons made with other schools, as in the use of standardized achievement tests?

3) Read pages 8-10

4) Comment II: Some teachers feel very uncomfortable with the idea of making judgments about others. Yet, that is part of a teacher's responsibility. The making of judgments is a little less odious if one has well defined objectives against which to measure progress and good information for making the judgments.

d. Non-paper-and-pencil Procedures.

1) Comment I: Teachers can secure much information by watching and listening to their pupils. It is impossible to store all of this information. Some is remembered by the teachers, some is recorded in some form, some is discarded, and, of course, some is forgotten. The major problem is determining which information secured from watching and listening is important enough to retain.

2) Questions

a) When one observes, what should be seen?

b) When one listens, what should be heard?

c) In what forms can important evalutive information be retained?

3) Read pages 10-14

4) Comment II: As we have noted previously, record keeping can be burdensome. The only recommendation that we can make with honesty is that you record only that important information that you cannot keep in mind. You will probably find that the first few anecdotal records that you make are time consuming. As with most behaviors, you can expect to write them more quickly as you gain experience in writing them. Notice that an inventory need not be strictly paper-and-pencil but may well include performance tasks, as modeling the addition algorithm for an exercise with an open-end abacus. Please keep in mind that pupils can also complete inventories and checklists. For a new teacher, administering such instruments provides information, admittedly rough information, in an economical manner.

e. Paper-and-pencil Instruments.

1) Comment I: Both attitude scales and tests are important sources of information for teachers. In all honesty, scales are rarely used. We recommend that some form

of attitudinal measure be secured at least two times a year. Tests should be designed to the requirements of the use to which the results will be put. Four are discussed on these pages.

2) Questions
    a) How can attitudes toward mathematics be measured?
    b) How does a criterion-referenced test differ from a norm-referenced test?
    c) Of what use are standardized tests?
    d) How does a diagnostic test differ from other forms of test?
3) Read pages 14-21
4) Comment II: Attitudinal measures from elementary school pupils are notoriously unstable. However, as a rule most elementary school pupils have favorable attitudes toward mathematics. One study showed math ranking second only to reading as the favorite subject of pupils in the intermediate grades. The study also reported that math was most often reported as "most liked" and "least liked." The items on the four kinds of tests discussed here are markedly similar. The important differences are noted in the discussion.

f. Planning the Test
1) Comment I: Again, we would recommend that you construct a terminal test as a part of your planning for teaching a unit. You may well have to change parts of it after teaching the unit. Constructing the test before teaching does force you to consider the terminal behaviors for which you are teaching.
2) Questions
    a) In what ways are well-stated objectives of value in constructing a test?
    b) How do you decide what kinds of test items to use on a test?
3) Read pages 21-23
4) Comment II: Beginning teachers typically make tests that are too long. Speed is rarely a criterion in math tests at the elementary level. Strive for a test that can be completed by virtually all pupils in the allotted time. Instructions are important. Give lots of your attention to them. We recommend that your outline of specifications use all five of the NLSMA categories. Any one test need not have items from all five categories.

g. Writing Test Items
1) Comment I: All of the common types of test items are appropriate for use in elementary school mathematics.
2) Questions
    a) Each type of item lends itself to measuring what kinds of learning?
    b) How should various types of items be arranged on a test?
3) Read pages 23-33
4) Comment II: If for no other reason than their being required to respond to various types of items on

standardized test, pupils should have the opportunity to respond to all types of items on teacher-made tests. That is not to say that all types should appear on each teacher-made test. More importantly, good test construction usually requires a variety of types. The multiple choice item is probably the most versatile, but it, too, has limitations. We recommend that for computational items that a plan be devised for awarding partial credits. Most instruction with computational algorithms should have some emphasis in addition to correct answers, which should probably receive greatest weight. You can find a much more detailed discussion in Bloom, Hastings, and Maudaus' _Handbook on Formative and Summative Evaluation of Student Learning_. One chapter shows application of principles of test construction to secondary mathematics. Many of the items are also suitable for use in elementary schools.

C. Review
1. Look back over the seven parts of B-2 above. Try to make one definitive statement that summarized the important information in each part.
2. Prepare any questions that you want your instructor to discuss with you.

D. Assignment
1. Immediate-- Locate a chapter in your pupil text that will lend itself to the use of all five types of items discussed on pages 25-33. Then construct the indicated number for each type of item:

Short answer	(4)
Multiple choice	(3)
Alternate choice	(5)
Matching	(2)
Essay	(1)

Be sure to include instructions for each type of item.

2. Long range-- Nothing specific. Tests should be a part of the materials that you submit for units to be taught.

Part Two

This part of the syllabus has three units. The first describes the development of basic concepts of number. The second is concerned with a sequence used in developing concepts of addition, the basic addition facts, and the standard algorithm for addition. The third part consists of similar development for subtraction.

Most of the instructional topics under this part are given greatest emphasis in the primary grades. Arrange to observe math classes in one primary grade for, say, three days. This will be a part of your directed observation for these three units. Keep in mind while observing that the emphases in this course are curricular sequences and instructional materials.

You can expect that the comments and supplementary readings in this part will be less extensive than in Part One.

Unit IV- Concepts of Number

A. Materials
  1. Professional textbook
    a. Heimer and Trueblood (H&T)
  2. Manipulatives (found in resource center)
    a. Unifix blocks
    b. Cuisennaire rods
    c. Open-end abacus
    d. Multi-base blocks

B. Study
  1. Directed observation- The materials to which you were referred in A. 2. of this unit are but samples from the vast number that are available. These are all commercial materials. No doubt, you can visualize similar materials that are homemade. In fact, materials familiar to children should be used in developing concepts of number. Bottle caps, popsicle sticks, dowel rods, graph paper, and such are examples of everyday referents for number. Take a closer look at the commercial materials to which reference is made.

The Unifix Blocks are examples of what are called discrete models of number. Some educators refer to them as digital models. The distinguishing quality of these is that the units can be observed and counted.

Take a double-handful and spread them out in front of you. Try, without counting, to assign a number to them. Now, count them. If you came within five or six of the exact number with your guess you are doing well. Most adults determine the number of a set like this by visualizing subsets of three, four, or five and adding to find the exact number.

Now, assemble the 100-tray. Join the blocks to make a "train". Place the train on the 100-tray. (We trust that the end of the train is at the same as your counting number for the set of blocks.) This tray is used in just this fashion: to confirm counts of objects. The tray also helps pupils develop a sense of "big" numbers, here related to length. But we are getting ahead of a teachable sequence. Push these blocks aside for now.

Make a train of three white blocks and four brown blocks. Place it on the tray. Note the number. Remove the train, but don't break it apart. Make a train with two whites and 5 browns. Place it on the tray. Note the number. Remove the train and place it beside the other train of whites and browns. Is one as long as the other? You have demonstrated that trains of the same length can be built from different numbers of blocks of different colors. This kind of exercise is the basis for a pupil's generalizing eventually that "a number has many names." It is obviously a readiness experience for basic addition facts (3+4=7, 2+5=7, eventually, 7= 3+4= 2+5. . . .)

One last task with the Unifix Blocks. Place the 10 x 10 tray before you. Take your double handful of blocks. Fill as many columns of the tray as you can with joined blocks, beginning on the left. How many columns did you fill? How many were left over? What number have you modelled?

It is always a good idea to remind your pupils to return their manipulatives to their proper position. We remind you to do the same thing.

The Cuisennaire Rods are examples of what are called continuous models of number. Some educators refer to them as analog models. The distinguishing feature is that the units cannot be observed. A number is assigned to a rod by comparing it witn a unit (almost always the white rod). Rather than being related to a count, number is related to measurement in the use of these rods. By the way, the white rod is a cubic centimeter. The rods are of some value in teaching metric units of measurement.

Let's start a bit differently with these. Starting with the orange rod, build a staircase. If the number "one" is assigned to the white rod, what number should be assigned to the orange rod? There is a pattern to the colors. Try to find it. (Hint: Use these groupings 1,7; 2,4,8; 3,6,9,). Refer again to your staircase. A rod of what color would have to be joined to each rod to make it as long as the one below it?

When pupils first begin working with these rods, they refer to them by color and use an interesting notation. Let's try it. Make a train of a red and a light green. Now find one rod as long as your train. Yes, the red and light green train is as long as the yellow rod. The notation is r=g+y. Make another train of two rods that is as long as the yellow. (Order is important.) The four trains in this set provides readiness for four of the six addition facts in the family of facts for five. (0 +5 and 5 + 0 are not represented, unless you want to consider the yellow rod as representing both.)

The rods can also be used to represent numbers greater than 10. Take a handful of rods of various colors. Make a train of them. Make another train consisting of only orange rods and one rod of another color (if necessary).

Cuisennaire rods can also be used to represent fractions. We shall do that in Unit IX. For now, return them to their container.

Both the Unifix Blocks amd Cuisennaire Rods, as you have used them, are referred to as proportional models. This means that each model of a number is that many times as big as the unit. For example, two orange rods and a dark green rod are 26 times as long as a white rod (the unit--one).

The open-end abacus is used for non-proportional models of numbers. Place the rods in the holder. The left rod(black) represents hundreds place, the middle rod (either blue or green, say blue) represents tens place and the right rod (green) represents ones place.

Pick up all of the green beads and place them on the green rod. Count them. How many? (18, why exactly 18?) An appropriate question to ask pupils at this point is "Can eighteen be written as a one-digit numeral?" Now, trade ten green beads for one blue bead and place the blue bead on the blue rod. You should now have one bead on the blue rod and eight beads on the green rod-- a non-proportional physical model for the number eighteen.

Many other materials are available that capitalize on the place value characteristic of our system for noting numbers. Other kinds of abacuses are available, especially the Oriental abacus and what is called the modern abacus. The modern abacus is closed at both ends and has either nine or ten beads fixed on each rod. Niether of these is useful as a model for standard computational algorithms.

Any three-digit number can be modelled with the open-end abacus. For a model of 372, place three black beads on the black rod, seven blue beads on the blue rod, and two green beads on the green rod. A zero in any place is represented by an empty rod.

The open-end abacus can also be used to model the renaming of numbers, as in using the subtraction algorithm. For example, 372 can be renamed to show 200 + 170 + 2 by placing two beads on the black rod, 17 on the blue and 2 on the green. Use the open-end abacus to represent numbers until you are confident that you can make a model of any number having three or fewer digits.

The multi-base blocks are proportional models for numbers in decimal and non-decimal bases. The blocks that we have available can be used as models for numbers in base two, three, four, five, and ten. The large cubes represent the third power of the base, the squares (or "flats") represent the second power, the strips (or "longs") represent the first power, and the smallest cubes represent the zero power or ones. Recall that a four digit number can be expressed as $(ax10^3) + (bx10^2) + (cx10^1) + (dx10^0)$ where 10 (referred to as one-zero, except in base ten) is the base number.

Let's make a model in base-three for this many objects: 1111111111 1111111111 1111111111 1111111111 1111111111 11. Count out that many small cubes. Make as many three-by-three-by-three cubes as you can. Then, make as many three-by-three squares as you can from the small cubes that are left. Then make as many three-by-one strips as you can from those that are left. Now, trade for connected pieces. If you did this correctly, you should have one large cube, two flats, two strips, and one small cube. Your $52_{(ten)}$ objects are modelled in base three as $1221_{(three)}$. To verify your accuracy, check in base ten: $(1x3^3) + (2x3^2) + (2x3^1) + (1x3^0) = 27+18+6+1=52$. Try enough more numbers in various bases until you feel confident with the blocks as models.

2. Comment

a. Crude concepts of number. Developmental studies of cognitive growth have revealed that even very young children have functional concepts of quantity. These exist before children develop the corresponding verbal concepts and formal number concepts. With the development of speech, these functional concepts are used for expressing quantitative ideas. In fact, some authorities refer to concepts like many, few, more, as many as, and same as crude concepts of number. Most curricula devote time to concepts of "few" and "many" before dealing with set equivalence and non-equivalence.

b. Approaches to basic concepts of number. The approach described on pages 53-57 of H&T is essentially what is called a cardinal approach. The alternative is called an ordinal approach. The basic difference is the function and place of rational counting (enumeration). The distinction is important enough for us to devote some time to it.

The most extreme of the cardinal approaches was found in the Scott-Foresman textbook series of the early and mid 60's. The authors summarize their approach in Charting the Course for Arithmetic, a professional textbook based on the series.

The approach can be thought of as progressing in these stages:

i. Crude comparisons-- As indicated above, these give emphasis to concepts of many and few.

ii. One-to-one correspondences-- Many experiences in matching objects (cups to saucers, erasers to pencils, chairs to children, etc.) or drawings of objects. The concepts are those related to equivalence (same, as many as, equal) and non-equivalence (more, fewer; greater, less.)

iii. Recognition of elemental groups-- identifying the number in sets having two, three and four members without counting.

iv. Recognition of larger groups-- Pupils are expected to identify the cardinal number of sets having as many as ten numbers, still without counting. They begin with exercises requiring them to identify sets of six, eight, and ten member. Their instruction guides them in recognizing subsets of two, three, four. Then they turn to sets of five, seven, and nine. Here the emphasis is on the idea of "one extra", as a set of four and one extra. This is consistent with what you as an adult probably did in your first experience with the Unifix Blocks. These exercises provide readiness for the basic addition facts and the concepts of odd and even numbers.

v. Counting-- Not until pupils can recognize sets having as many as ten members do they use rational counting. Charts like the one on page 56 are developed with pupils. Heavy emphasis is given to the idea of "a number and one more," as four and one more is five.

The impetus for the cardinal approach came from two sources. (1) Educators recognized that many youngsters could say the number names in order, that is they could count by rote, but could not enumerate a set of objects. Moreover, several research studies had found that many youngsters had little "number sense", even of sets having fewer than ten members. (2) Mathematicians have given serious study to the development of number systems. One set of postulates is markedly similar to this approach.

The writers of this syllabus are not familiar with any current textbook series that currently delay rational counting this long. Several series use the cardinal approach through, say five and then shift to the ordinal approach. Even the series Sets and Numbers, which includes many exercises like the one found at the top of page 58, includes instructions for pupils to count (rationally) to find the number of members of a set.

In contrast to the cardinal approach, the ordinal approach begins with counting. It, too, can be thought of as progressing in stages, though there is overlap among the stages. The stages are these:

i. Rote counting-- This is nothing more than saying the number names in correct order. Most entering kindergarteners can count to somewhere between ten and twenty, some to a hundred or more.

ii. Rational counting-- This is nothing more than assigning one counting number to each object a set. It is the one-to-one correspondence as illustrated on page 57.

iii. Identification- This is the stage that is most often omitted. Speculation is that it is this omission that has led to the lack of number sense. At this stage pupils are supposed to recognize the number of members in a set without counting. It is best done by using quick exposure of cards similar to the one illustrated on page 64. One reason for this omission is that many of the exercises are assigned to first graders as seatwork, especially when other pupils are in reading groups. Pupils have time to count the members in this circumstance.

iv. Reproduction-- Pupils are instructed to draw six balls, five dolls, ten toys, and such.

v. Comparison -- This stage finds exercises in which pupils are to identify, as by crossing out, the larger set or the smaller set or to record the number of two sets and relate them with the greater than (>), less than (<), or equal to (=) symbols.

vi. Grouping-- Pupils are asked to recognize the number of subsets of some superset, as subsets of four and three in a set having seven members.

We have devoted this much space to these two approaches primarily so that you will not misuse the instructional materials you have available to use in your teaching. No clear cut research evidence gives support to either approach over the other.

C. Importance of counting. The flow chart on page 53 stops with the box, "order of cardinal numbers." The importance of counting forward by ones, as in enumerating a set of objects is rather obvious. Counting backwards by ones and counting forwards and backwards by multiples are less obvious. All relate to the basic operations of arithmetic. Counting forwards by ones is related to additon, backwards by ones to subtraction, forwards by multiples to multiplication, and backwards by multiples to division.

Pupils do not need the same proficiency in all kinds of counting. They do need experience with all kinds. Proficiency in counting at least nine forward or backward from any two-digit number is desirable. Counting forward by multiples of two, three, five, and ten has rather high social utility. Even though pupils may develop little proficiency in counting backward by multiples, the idea of serial subtraction (75-8=67, 67-8=59, 59-8=51, etc.) is important.

One of the slogans of math educators is this: counting should be rational. However, there are exceptions- Multiple counting, even though by rote, does have value in developing concepts related to the operations.

Facility in counting does take a lot of time. Fortunately, practice is an excellent "time filler". For example, you can say to a class, "Let's see if we can count to one-hundred by fives before the bell rings." A subtle reminder is the display of number tables similar to the one at the bottom of page 67. For example, the table could have eight columns. Each column then gives the multiple counting order by eights from a given number.

d. Place value. We observed in Unit II that many computational errors stem from a poor understanding of place value. The development of the concepts of place value typically is concurrent with other learnings and is easily neglected. For example, numbers in the second decade (10-19 or 11-20) are commonly developed along with the basic addition and subtraction facts.

Pupils need many experiences where they use proportional representations, like the one at the top of page 59, at both the concrete and semi-concrete levels. They need to progress from proportional representations to non-proportional representations, like the one at the bottom of page 59.

Do keep in mind that color-coding is quite useful in drawing pupils' attention to places in place value systems. Color-coding is most valuable with nonproportional models, like abacuses and place value pockets.

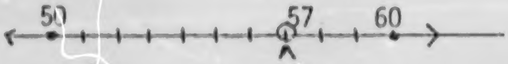
The instructional vignettes that we used in Unit I had number periods as the topic. For some reason many pupils have difficulty reading large numbers. Do emphasize that repeating pattern of ones-tens-hundreds as shown in the diagram on page 68. Color coding is also useful in making period charts of this kind.

As a sidelight, you should be aware that our way of marking off number periods and naming them is a convention. Other cultures use other ways.

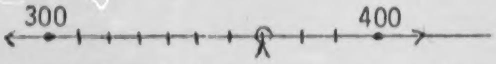
A very important part of work with place value is rounding numbers to the nearest ten, hundred, thousand, and so forth. Not only does this ability have reasonably high social utility, it is useful in other parts of the curriculum. For example, rounding factors in multiplication to yield an estimate of the product is something that we shall be emphasizing in Unit VII. Most textbooks are almost void of practical applications. Yet, applications abound in everyday life. Draw on them.

The number line is an excellent way of introducing the idea of "closer to." To illustrate take these two examples.

Is 57 closer to 50 or 60?



Is 365 closer to 300 or 400?



Such visual means are to be used before introducing the usual rule.

The authors of the text give some attention to non-decimal systems of notation. These were introduced into the elementary school curriculum within the last 20 years. The hope that work in non-decimal bases would result in greater understanding of base-ten and numeration systems in general has not been supported by the research. However, good reasons still exist for work with non-decimal systems. They are important in the discipline of mathematics. The content provides a vehicle for review of other learnings in a rather unobtrusive manner.

C. Review
1. Refer to page 52 in H & T. Formulate questions based on each objective. Then, answer them to your satisfaction.
2. Select a math text for the primary grades. Thumb through it in an effort to recognize the application of the kinds of things that have been studies in this unit.
3. Prepare any questions that you want your instructor to discuss with you.

D. Assignment
1. Immediate - respond to exercises 1, 2, 3, 6, 7, 10, 11, 13, 14, and 19 on pages 70-72.

Unit V. Addition

A. Materials
1. Professional textbook
    - a. Heimer and Trueblood (H & T)
    - b. Ashlock
2. Manipulatives and Task Sheets
    - a. Cuisennaire rods
    - b. Balance scale
    - c. Open-end abacus
    - d. Chip-trading activities
    - e. Checkers
3. Pupil textbook of your choice

B. Study
1. Directed observation
    - a. Complete the task sheets, which are attached to this unit. They are intended to familiarize you with some of the manipulative materials that can be used in teaching addition.
    - b. Refer to Figure 4.5 on Page 89. Think back on the observation that you did in a primary-grade class as perparation for Part Two of this course. Try to identify which, if any, of the topics were being studied by pupils.
2. Basic reading assignment
    - a. Read the synopsis and objectives on pages 76 and 77 of H & T.
    - b. Study pages 135, 19-23, 53-57, and 83-87 in Ashlock.
    - c. Read pages 79-114 of H & T.
3. Comment
    - a. Meanings of addition. Though sophisticated adult mathematicians may think of addition without reference to an action on sets of objects, children should. That action is the joining, real or imagined, of two sets of objects. The discipline of mathematics has its roots in the real world. Pupils should relate the addition of numbers to the joining of sets of objects taken from their real worlds.

        Set notation, like that found on pages 79-80, was introduced into the elementary curriculum in the mid-sixties. One reason is that it enables one to make a record of actions on objects as opposed to operations on numbers. For example, $\{\square, \triangle, \triangledown\} \cup \{0\}$ $\{0, \triangledown, \square, \triangle\}$ is a record of the joining of two sets to yield a third set, while $3 + 1 = 4$ is a record of adding two numbers to yield a third number.

        Regardless of whether set notation is used, pupils need to learn to relate the joining of sets of objects to the addition of numbers. The progression from concrete (as bottle caps) to semi-concrete (as pictures) to abstract (numerals only) is recommended.

        Note that reference was made to real or imagined actions. Most beginning experiences involve actions: one has a number of objects, those objects are joined by a number of other objects, and the join of the two sets is the total ($n_1 + n_2 = N$). Sometimes, though, the action has to be imagined. For example, I have 4 pencils in my desk and one or top of it for a total of 5. One has to "imagine" the joining of the two sets.

Basic additon facts. The use of the noun "facts" is unfortunate. However, we are stuck with the term. The use is unfortunate in that the basic addition facts should be taught as principles, which they are. They can be deduced from other information. For example, and with lots of ommissions:

$$\{\triangle, \square, \triangledown\} \cup \{o\} = \{\square, o, \triangle, \triangledown\}$$

$$n\{\triangle, \triangledown, \square\} + n\{o\} = n\{\square, o, \triangle, \triangledown\}$$

$$3 \quad + \quad 1 \quad = \quad 4$$

This kind of development should be available to learners even if set notation is not used.

Do note that the basic addition facts have addends that are one-digit numerals. Other number combinations, such as 10 + 10 = 20 and 12 + 12 = 24, can be learned as "facts" or facts, but they are not basic addition facts.

The evidence is quite clear that once pupils learn a basic addition fact, like 4 + 2 = 6, that knowledge of the other member of the commutative pair, like 2 + 4 = 6, can almost be assumed. This is one reason that you will find so much emphasis on commutative pairs when pupils first begin to study the facts.

H & T make suitable note of the use of the associative property in developing the facts having sums of 11 - 18. It is also useful in reconstructing a fact if it is forgotten. For example, a pupil may forget the sum of 6 + 8 but remember that 6 + 6 = 12. So, he should have experiences that would cause him to think "8 is two more than 6." I can rename 8 as 6 + 2. Now 6 + (6 +2) = (6 + 6) + 2. 6 + 6 = 12, 13, 14. Eureka! 8 + 6 = 14.

The symbolism of the addition facts needs a little attention. 4 + 2 = 6 should be read "four plus two equals six." Save the "and" and "are" for "4 discs and 2 discs are 6 discs." It is also appropriate to say from time to time that "four plus two is another name for six." This lays a foundation for later study involving renaming. Early work with the facts is almost always in horizontal notation. The primary reason is that the sentence form utilizes the left-to-right progression that kindergarteners and first graders use in reading other kinds of subject matter.

Other than their comments about using more than two addends when developing the associative property H & T make no reference to what is commonly called column addition, that is, more than two addends. Column addition is usually introduced after pupils have had experience with the facts in vertical notation, called the computational form. Column addition can be used to provide novelty to practice on the facts, which can become pretty boring to children. Do observe that 4 + 1 + 3 + 9 = ☐ calls for only basic facts while 9 + 3 + 4 + 1 = ☐ calls for other learnings.

If pupils are to do well in using an addition algorithm, they must have mastery of the basic additon facts. That mastery takes practice. Practice can be pure drudgery for many youngsters. Practice pages containing nothing but basic number questions have a proper place but so do novel practice exercises. The novel exercises may be in the form of games, coloring areas of drawings containing number questions, and cross number puzzles. Much "sugar coating" is recommended. Practice pages provide information that pupils can use in charting their progress towards mastery. First, in terms of accuracy. Later, in terms of speed. A reasonable goal for speed is 100 facts in about four or five minutes. Oral practice exercises are also recommended, especially for that child who is an auditory learner. When conducting oral practice insist that a pupil respond to "Four plus eight equals what?" by saying "Four plus eight equals twelve." In other words, have your pupils practice the statement you expect them to learn.

Some special problems arise for some pupils in learning the facts. Even in these days of meaningful instruction, the facts with zero as an addend give trouble. Give special emphasis to sets having no members. Difficulties with unknown addends were mentioned in Unit I. Counting can cause problems if not taught properly. The hands are about the handiest calculators a pupil has handy. The biggest problem occurs when a basic number question like $8 + 4 = \square$ is being calculated. The pupil lays out four fingers, touches the little finger and says "eight", touches the ring finger and says "nine", touches the middle finger and says "ten", touches the index finger and says "eleven," and then states proudly,"Eight plus four equals eleven."

The flowchart on page 85 of H & T shows the basic additon facts as prerequisite to beginning study of the addition algorithm. Yet, you will likely find some pupils who seem unable to memorize all 100 of the facts. Some educators insist that mastery of the facts be gained before undertaking study of the algorithm. Other educators would have these pupils proceed if other prerequisites have been mastered. We recommend the latter. Further practice can become deadening. Practice is inherent in the use of the algorithm. A common substitute for complete mastery of the facts is facility in the use of an addition table.

c. Addition algorithm. In broad terms an algorithm (or algorism) is a step-by-step procedure for solving a problem or accomplishing some end. As used in arithmetic an algorithm consists of a finite number of steps involving repetitions of an operation.

Have you ever given thought as to why we teach algorithms, like addition algorithms? An algorithm enables one to rely on memorized information, the basic facts, to find a solution rather than having to memorize hundreds of thousands of facts like 365 + 48 = 413.

Several addition algorithms are available. The one used by most adults, and developed on pages 83-89 in H & T, is referred to as a standard algorithm. If you are like most adults who have some facility with numbers, you probably improvise on the standard algorithm you were taught. Watch for pupils who do this kind of improvising. It is a sign of approaching maturity in the use of addition.

Reference is somtimes made to these terms: immature algorithm and intermediate algorithm. Immature algorithms are most commonly found in discovery lessons and are usually created by pupils. For example, suppose that a discovery episode motivates a number question like 12 + 5 = ☐ and then that pupils are instructed to find the answer "in any way you can." One pupil may count 12, 13, 14, 15, 16, 17. Another may think "Twelve is five plus five plus two and five more: 5, 10, 15, 16, 17. Still another may make a set of 12 tally marks, another set of 5, ring 10 of them, count the remainder and arrive at 17. All of these are evidences of immature algorithms. Intermediate algorithms may evolve from pupils' immature algorithms or, more commonly, be a part of a teacher's explanation. The first two of the three-step processes found on pages 83-84 and 86-89 are intermediate algorithms. They record more of the thought processes than the standard algorithm. A word of caution is in order. Though a teacher may use examples like

$$\begin{array}{r} 13 \\ +6 \\ \hline \end{array} \longrightarrow \begin{array}{r} 10 + 3 \\ + 6 \\ \hline 10 + 9 \end{array} = 19 \qquad \text{and} \qquad \begin{array}{r} 26 \\ +3 \\ \hline 9 \\ 20 \\ \hline 29 \end{array} \longrightarrow \begin{array}{r} 20 + 6 \\ + 3 \\ \hline \end{array}$$

as parts of explanations, the wisdom of having pupils do the same thing is questionable. However $\begin{array}{r} 26 \\ +3 \\ \hline 9 \\ 20 \\ \hline 29 \end{array}$ is an important intermediate step. Recording tens in one color pencil or chalk and ones in another, then showing the partial sums (9 and 20) in the same colors is of value at this stage.

The sequences that are explained on pages 83-87 are found in most textbook series. Two cautions in their use must be made. (1) The sequence engenders errors like $\begin{array}{r} 66 \\ +28 \\ \hline 1\overline{14} \end{array}$ and (2) the work with sums of multiples of ten is not to be ignored. The errors can be reduced by the teacher modeling and insisting on a right to left sequence of computations with questions like these in solving for 46 + 53 = ☐ : "Six ones plus three ones equals how many ones?" "Can that be written with a one-digit numeral?" In what place do we write the sum of the ones?" "Four tens plus five tens equals how many tens?" Can that be written with a one-digit number?" "In what place do we write the sum of the tens?"

The sequence of concrete to semi-concrete to abstract applies to the examples found on pages 83-87 and some work with semi-concrete representations to those found on page 88. It is easy for teachers to move too quickly to the abstract level. As a general rule, pupils should be permitted to use concrete and semi-concrete representations even when the emphasis is on securing responses at the abstract level.

The second box in Figure 4.4 describes a topic to which special attention must be given because of its value in adding long columns of numbers. In fact, examples of this kind (24 + 8 = 32, 34 + 7 = 41, 69 + 7 + 76, etc.) are given a special name, bridging. Persons who have some facility with numbers reach a stage where they do not consciously "carry" to the next decade. Rather, their thought process is something like "four plus 8 equals twelve, thirty-two."

As was mentioned in Unit IV, pupils should have a great deal of experience in estimating sums <u>before</u> computing. The reason for underscoring "before" is that pupils frequently force their estimates to agree with their sums after adding. Pupils should also be required to check a goodly number of sums, particularly when doing column addition. Adding up after adding down is most commonly recommended.

Alternate algorithms are available. Some are for less capable youngsters who just have difficulty with keeping all the steps in mind. Others are suitable for enrichment.

The algorithm illustrated below is called a low-stress algorithm. Note that it calls for no bridging, no mental storing of partial sums, and limited knowledge of place value

```
  ②          ②
   2 4        9 8
   4          11
   8 8        1 1
  12 12       2 9
   5 6        4 6
   7 8        6 15
   9 2        2 5
  16 10       8 10
   8 6        8 9
  14 6       16 9
 ------     ------
  3 4 6      2 6 9
```

The two algorithms below are presented with little explanation; try to figure them out.

Front-end	Zig-zag	(think)
476	476	476
398	398	776, 866, 874
635	635	1474, 1504, 1509
1~~3~~~~9~~9 ~~4~~0 5	←	

C. Review

1. Refer to pages 76-77 in H & T. Formulate questions based on each objective. Then, answer them to your satisfaction.
2. Select a pupils math text for a level of interest to you. Thumb through the parts, including review sections, that deal with addition. Identify where the topics fall on the comprehensive flow chart. As preparation for the next unit, observe whether subtraction is juxtaposed with addition. It usually is.
3. Prepare any questions that you want your instructor to discuss with you.

D. Assignment

1. Immediate -- Respond to exercises 2, 5, 6, 7, 8, 10, 12, 14, 15, 19 on pages 110-114 of H & T.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
5,7,1

Task Sheet 2.1.2.1 Models for the Basic Addition Facts

Prerequisite:
Materials: Cuisennaire Rods
Problem: How can Cuisennaire rods be used as real world models of the basic addition facts?
Tasks:

1. Remove the rods from their container. Place them in a pile.

2. Complete this table by making a "staircase" using one rod of each color.

1	white	w	____	dark green	d
2	red	r	____	black	k
____	light green	g	____	brown	n
____	purple	p	9	blue	b
____	yellow	y	10	orange	o

3. Place any two rods from the first column above end-to-end. This is called a "train." Trains may consist of more than two rods.

4. Make a train of a light green rod and yellow rod. Locate one rod that is as long as the train.
What color is that rod? ____________

5. What number is assigned to the light green rod? ____________
What number is assigned to the yellow rod? ____________
What number is assigned to the other rod? ____________

6. You have made a model of an addition fact: 3 + 5 = 8.
How would you show 5 + 3 = 8? (Use the letters from the table in an equation: y + _?_ = _?_) ____________

7. Return your rods to the stack.

8. You have shown that 8 is the sum for two basic addition facts.
Now make a "staircase" with the brown rod at the bottom.

9. Make a train as long as the brown rod on each level of the staircase.
How many basic addition facts are modeled with trains of two rods? ____________

10. Place another brown rod at the top. The two brown rods are models of what other basic addition facts? ____________ ____________

11. How many basic addition facts have 8 as their sum? ____________

12. Place the top brown rod below the bottom brown rod. Place each train of the same two colors on successive rows below the brown rods.
Rods of what color remain above the brown rods? ____________

13. The pairs of the successive rows, including the browns, are models of cummutative pairs. How many commutative pairs have 8 as their sum? ____________

14. You may want to try similar tasks for other basic addition facts having the same sums. Sums greater than 10 have to be modeled with two rods, orange and one other rod. Three interesting patterns appear: the sums having "doubles," the number of commutative pairs, and the number of facts related to each sum.

15. Return the rods to their container.

Task Sheet 2.1.2.2 Models for the Basic Addition Facts

Prerequisite:
Materials: Balance Scale
Problem: How can the balance scale be used to model addition facts?
Tasks:

1. Make sure that the scale is in balance.

2. Place 1 weight on the left-arm 5-peg.
Place 1 weight on the left-arm 4-peg.

3. Place 1 weight on a right-arm peg that will make the scale balance.
On what peg did the scale come into balance? ____________

4. You have shown that 5 + 4 = □ ____________

5. Remove the weights from the left-arm pegs.
Place them on the left-arm 3- and 6-pegs.
Does the scale balance? ____________

6. Now, let's try some sums in the second decade.
Place 1 weight on the left-arm 7-peg and one weight on the 5-peg.
Is 7 + 5 > 0? ____________
Is 7 + 5 > 10? ____________

7. Keep the weight on the right-arm 10-peg.
Place another weight on a right-arm peg that will make the scale between.
On what peg did you place the weight? ____________

8. You have shown that 7 + 5 = 10 + □ ? ____________

9. What is the standard name for 10 + 2? ____________

10. So, you have demonstrated that 7 + 5 = □ ? ____________

11. Try some other basic facts, including the zero-facts, like
4 + 0 = 4 and 9 + 0 = 9.

12. Return the weights to their storage pegs.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
5,7,3

Task Sheet 2.1.3.1 Models for the Addition Algorithm

Prerequisite: 2.1.2.1
Materials: Cuisennaire Rods
Problem: How can Cuisennaire rods be used as real world models of the standard addition algorithm?

Tasks:

1. Remove the rods from their container. Place them in a pile.
2. Make a train of 1 orange rod and 1 dark green rod (o + d). What number is associated with this train? ________
3. Make a train of 2 orange rods and 1 light green rod. What number is associated with this train? ________
4. Join the two trains with the orange rods on the left and the green rods on the right.
5. Trade the 1 light green rod and 1 dark green rod for one rod as long as the two greens.
6. Your train has how many orange rods? ________
   The orange rods represent what number? ________
7. The blue rod represents what number? ________
8. You have shown that 16 + 23 = ☐ ? ________
9. You may want to try the addition of other one- and two-digit numbers without renaming. Let's now find how to represent renaming.
10. Make a train of 2 orange rods and 1 purple rod. This train represents what number? ________
11. Make a train of 2 orange rods and 1 brown rod. This train represents what number? ________
12. Join the two trains, keeping the orange rods together.
13. Trade the purple and brown rod for a train made up of an orange rod and one other. This train represents what number? ________
14. How many orange rods in 12 above? ________
    They represent what number? ________
15. How many orange rods altogether?
16. What is the value of the "other" rod? ________ ________
17. You have shown that 24 + 28 = ☐ ? ________
18. You may want to try other addition questions.
19. Return the rods to their container.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
5,7,4

Task Sheet 2.1.3.2 Models for the Addition Algorithm

Prerequisite:
Materials: Open-end abacus
Problem: How can an open-end abacus be used to model the standard addition algorithm?
Tasks:

1. Assemble the abacus by placing the black rod in the left position, the two green rods in the middle position, and the two blue rods in the right position. Place the counters in a pile.
2. Place 3 green counters on the green rod and 7 blue counters on the blue rod. What number have you represented? ____________
3. Place your finger across the counters and touching the rods. Then place 1 green counter and 6 blue counters on their matching rods. What number have you represented? ____________
4. Let the counters above your fingers fall to join those below. What operation is being modeled? ____________
5. How many blue counters? ____________
6. Trade 10 blue counters for 1 green counter. How many green counters altogether? ____________
7. What is the value of the green counters? ____________
   What is the value of the blue counters? ____________
8. What is the sum of 37 and 16? ____________
9. You may want to model some other additions.
10. Return the rods and counters to their containers.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
5,7,5

Task Sheet 2.1.3.3 Models for the Addition Algorithm

Prerequisite:
Materials: Chip Trading Activities Kit
Problem: How can trading chips be used to model the addition algorithm?
Tasks:

1. Inspect the pictures of materials on page 2 of Activity Book I.
   Place a chip till in front of you.
   Identify the other materials that are in the box.
2. Complete Activity I-15.
3. Complete Activity I-16.
4. Complete Activity I-17.
5. Return all materials to their container.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
5,7,6

Task Sheet 2.2.2.1 Models for Adding Signed Numbers

Prerequisite:
Materials: Set of American Checkers
Problem: How can the addition of integers be modeled with checkers?
Tasks:

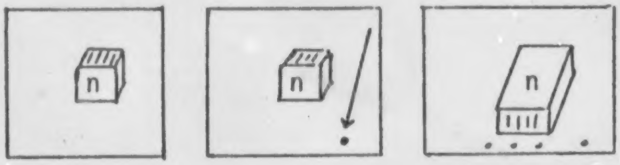
1. Each black checker has a value of +1.
   Each red checker has a value of -1.
2. Make a pile of five black checkers.
   What is the value of the pile? + ________
3. Make a pile of seven red checkers.
   What is the value of the pile? - ________
4. Join the reds to the blacks and mix them all up.
   What operation have you represented? ________
5. Make as many two-stacks, consisting of one red and one black as you can.
   How many red checkers are left over? ________
   What is their value? - ________
6. You have modeled +5 + -7 = ☐ ________
7. Model these:
   +5 + +7 = +12
   -5 + +7 = +2
   -5 + -7 = -12
8. Return the checkers to their container.

Unit VI. Subtraction

A. Materials
  1. Professional textbooks
    a. Heimer and Trueblood
    b. Ashlock
  2. Manipulatives and Task Sheets
    a. Unifix Blocks
    b. Balance scale
    c. Place value cans
    d. Checkers

B. Study
  1. Directed observation
    a. Complete the task sheets, which are attached to this unit. They are intended to familiarize you with some of the concrete level materials that can be used in teaching the subtraction facts and subtraction algorithms.
    b. Think back on the observation that you did in preparation for Part Two of this course. In your own mind, identify which, if any, subtraction topics were being studied.
  2. Basic reading assignment
    a. Read the synopsis and objectives on page 120 of H & T
    b. Study pages 135-136, 24-28, 57-61, and 88-94 in Ashlock
    c. Read pages 121-138 of H & T
  3. Comment
    a. Meanings of subtraction. Though defined as the inverse of addition in mathematics, subtraction is related to two other physical situations in the real world. They are referred to as the "take-away" and comparative interpretations. These diagrams vivify the differences for $4 - 1 = \boxed{3}$.

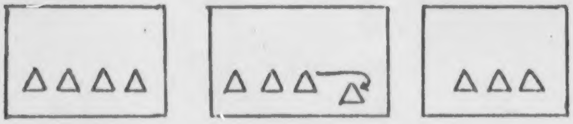
Additive



$n + 1 = 4$

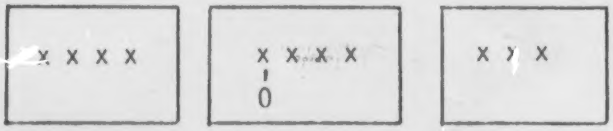
$4 - 1 = 3$

Take-away



$4 - 1 = 3$

Comparative



$4 - 1 = 3$

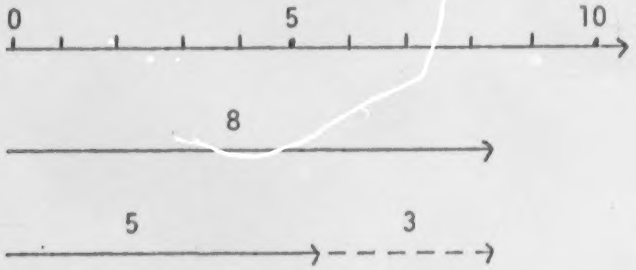
Pupils must have real world experiences with all three interpretations if they are to become facile in translating real world subtractive situations into subtraction sentence. This, of course, is a very important part of applying mathematical learnings.

Traditionally, pupils have been introduced to subtraction through take-away situations. During the early days of the so-called new math, efforts were made to draw a closer relation to addition by informing pupils that sentences like $\square + 4 = 5$ and $1 + \square = 5$ could be solved by subtraction, that is, $\square + 4 = 5$ can be stated as $5 - 4 = \square$ and $1 + \square = 5$ can be stated as $5 - 1 = \square$. We have already noted the problems that many first graders have with an unknown addend.

Most curricula now again introduce subtraction with take-away situations. The relationship between addition and subtraction can be made with actions on objects: addition as joining, subtraction as separating. Comparative situations are typically introduced second. Additive situations are commonly used in review at abstract levels. It is common practice to organize some mastery lessons around what are called related facts. Related facts for five are these:

$$\begin{array}{r} 4 \\ +1 \\ \hline 5 \end{array} \quad \begin{array}{r} 1 \\ +4 \\ \hline 5 \end{array} \quad \begin{array}{r} 5 \\ -1 \\ \hline 4 \end{array} \quad \begin{array}{r} 5 \\ -4 \\ \hline 1 \end{array}$$

You need to be careful when selecting materials to model subtraction. Continuous models, like Cuisennaire rods or number lines, do not lend themselves well to the take-away interpretation. Notice the similarity between the number lines for $6 + -2 = \square$ on page 91 of H & T and $10 - 4 = \square$ on page 127. Upper grade teachers often have to do some "unteaching" when using the number line for representing addition of integers. The number line can be used to show the comparative and additive situations. It is used in this way for $8 - 5 = \square$:



b. Basic subtraction facts. A one-to-one correspondence exists between the basic addition facts and basic subtraction facts: $12 - 8 = 4$ and $4 + 8 = 12$, $12 - 4 = 8$ and $8 + 4 = 12$, etc. Fine differences exist among textbook series in the manner in which they introduce the basic subtraction facts. All introduce addition before subtraction. The number of facts in a unit and the physical separation between related addition and subtraction facts create the differences.

Subtraction facts with minuends in the second decade (11-18) seem to be a little more difficult for some pupils. However, this is not a major problem, especially if a close relationship between addition facts and subtraction facts is established.

You will recall that the first error pattern (S-W-1) in Ashlock was the result of subtracting the "little" number from the "big" number regardless of its position in the minuend or subtrahend. For this reason, a pupil should have some practice exercise items like $2 - 7 = \square$. A proper response is "We can't find the answer now" or "There is no answer in the set of whole numbers." This, of course, relates to the closure property (or lack of it) in the set of whole numbers.

As with the addition facts, mastery of the subtraction facts is essential to the successful use of computational algorithms for subtraction. The same recommendations apply. Can you guess what this pupil is doing: $8 - 5 = 4$?

We recommend that pupils first read mathematical sentences like $8 - 5 = \boxed{3}$ as "Eight minus five equals three." Many teachers have pupils say "take-away." However, this practice is a source of confusion when pupils encounter situations that are obviously not take-aways. Eventually, you will want pupils to be familiar with other ways of asking the subtraction question: Some are five from eight, eight less five equals what, five and what are eight, eight remove five leaves what, and eight is how many more than five.

c. Subtraction algorithm. As with the addition algorithm, a common progression is from immature to intermediate to abstract algorithms. The immature algorithms, again, are most commonly found in exploratory (discovery) episodes.

The parentheses in the intermediate algorithms are a source of confusion to pupils. We recommend that they not be used. Rather that two subtraction symbols be used, if at all, even though some mathematical niceties are violated.

$$\begin{array}{r} 58 \\ \underline{-21} \\ \end{array} \longrightarrow \begin{array}{r} 50 + 8 \\ \underline{-20 - 1} \\ 30 + 7 \end{array} = 37 \qquad \begin{array}{r} 95 \\ \underline{-28} \\ \end{array} \longrightarrow \begin{array}{r} 80 + 15 \\ \underline{-20 - 8} \\ 60 + 7 \end{array} = 67$$

The algorithm shown in H & T is called the decomposition method. For many years an algorithm called the equal-addends method was taught in schools. Pupils who have been helped at home may use it in your classes. Rather than "decomposing," a number is added to both the minuend and subtrahend. Here are three solutions using the equal-addends method.

$$\begin{array}{r} 6\ \overset{14}{\cancel{4}} \\ -\ \underline{\overset{3}{\cancel{2}}\ 8} \\ 3\ 6 \end{array} \qquad \begin{array}{r} 2\ \overset{14}{\cancel{4}}\ \overset{15}{\cancel{5}} \\ -\ \underline{\overset{2}{\cancel{1}}\ \overset{6}{\cancel{5}}\ 9} \\ 8\ 6 \end{array} \qquad \begin{array}{r} 4\ \overset{11}{\cancel{1}}\ \overset{13}{\cancel{3}}\ 9 \\ -\ \underline{\overset{4}{\cancel{3}}\ \overset{7}{\cancel{6}}\ 4\ 1} \\ 4\ 9\ 8 \end{array}$$

The research evidence, which is quite old, shows greater accuracy in the use of this method. However, it is more difficult to teach with meaning. It is also shown on page 141 of H & T.

The value of concrete and semi-concrete models is considerable. The task cards (3.1.3.1. and 3.1.3.2.) for the place value cans demonstrates a way of moving from proportional to non-proportional models.

As with the addition algorithm, children should estimate many differences and check many answers. The most efficient check is to add the difference and subtrahend.

A low stress algorithm is available for subtraction. The essential difference between it and the standard algorithm is that a user performs like steps at one time rather than comparing top number and bottom number, renaming, subtracting, etc. The steps in solutions using the algorithm are shown below:

$$\begin{array}{r} 8\ 4\ 7\ 2 \\ -\underline{6\ 6\ 7\ 3} \\ \\ \end{array} \quad \begin{array}{r} 8\ 4\ 7\ 2 \\ 6\,{}^{1}2 \\ -\ \underline{6\ 6\ 7\ 3} \\ \end{array} \quad \begin{array}{r} 8\ 4\ 7\ 2 \\ 3\,{}^{1}6\,{}^{1}2 \\ -\ \underline{6\ 6\ 7\ 3} \\ \end{array} \quad \begin{array}{r} 8\ 4\ 7\ 2 \\ 7\,{}^{1}3\,{}^{1}6\,{}^{1}2 \\ -\ \underline{6\ 6\ 7\ 3} \\ \end{array} \quad \begin{array}{r} 8\ 4\ 7\ 2 \\ 7\,{}^{1}3\,{}^{1}6\,{}^{1}2 \\ -\ \underline{6\ 6\ 7\ 3} \\ 1\ 7\ 9\ 9 \end{array}$$

When one or more zeros appear in the minuend a special rule is applied. The steps in a solution appear below:

$$\begin{array}{r} 7\ 0\ 0,\ 5\ 0\ 4 \\ -\ \underline{2\ 6\ 5,\ 9\ 1\ 8} \\ \\ \end{array} \quad \begin{array}{r} 7\ 0\ 0,\ 5\ 0\ 4 \\ \\ -\ \underline{2\ 6\ 5,\ 9\ 1\ 8} \end{array} \quad \begin{array}{r} 7\ 0\ 0,\ 5\ 0\ 4 \\ 4\ \ {}^{1}4 \\ -\ \underline{2\ 6\ 5,\ 9\ 1\ 8} \end{array} \quad \begin{array}{r} 7\ 0\ 0,5\ 0\ 4 \\ 6\quad {}^{1}4\ \ {}^{1}4 \\ -\ \underline{2\ 6\ 5,9\ 1\ 8} \end{array}$$

$$\begin{array}{r} 7\ 0\ 0,\ 5\ 0\ 4 \\ 6\ 9\ 9,\,{}^{1}4\ 9\,{}^{1}4 \\ -\ \underline{2\ 6\ 5,\ 9\ 1\ 8} \\ \\ \end{array} \quad \begin{array}{r} 7\ 0\ 0,\ 5\ 0\ 4 \\ 6\ 9\ 9,\,{}^{1}4\ 9\,{}^{1}4 \\ -\underline{2\ 6\ 5,\ 9\ 1\ 8} \\ 4\ 3\ 4,\ 5\ 8\ 6 \end{array}$$

Some algorithms are available for enrichment. The steps in three of them are shown below:

$$\begin{array}{l} \text{Front end} \\ \text{(Provident)} \\ \\ \\ \end{array} \begin{array}{r} 4\ 8\ 9 \\ -\underline{2\ 9\ 9} \\ \\ \\ \end{array} \quad \begin{array}{r} 4\ 8\ 9 \\ -\underline{2\ 9\ 9} \\ 2 \\ \\ \end{array} \quad \begin{array}{l} 4\ 8\ 8 \\ -\underline{2\ 9\ 9} \\ \cancel{2}\ 9 \\ 1 \end{array} \quad \begin{array}{l} 4\ 8\ 8 \\ -\underline{2\ 9\ 9} \\ \cancel{2}\ \cancel{0}\ 9 \\ 1\ 8 \end{array}$$

Complement of ten

$$\begin{array}{r}3\,6\,2\,1\\-\,\underline{1\,8\,1\,4}\end{array}\qquad\begin{array}{r}3\,6\,2\,1\\-\,\underline{1\,8\,1\,4}\\2\,\bar{2}\,1\,\bar{3}\end{array}\qquad\begin{array}{r}3\,6\,2\,1\\-\,\underline{1\,8\,1\,4}\\1\,8\,0\,7\end{array}$$

Complementary

$$\begin{array}{r}6\,0\,3\,5\\-\,\underline{2\,8\,4\,7}\end{array}\quad 2847+7153=10{,}000\quad\begin{array}{r}6\,0\,3\,5\\+\,\underline{7\,1\,5\,3}\\\cancel{1}\,3\,1\,8\,8\end{array}$$

C. Review

1. Refer to page 120 in H & T. Formulate questions based on each objective. Then, answer them to your satisfaction.
2. Again, select a pupils' math text for a level of interest to you. Thumb through the parts, including review sections, that deal with subtraction. Identify where the topics would fall on a flow chart showing the sequence of subtraction topics.
3. Prepare any questions that you want your instructor to discuss with you.

D. Assignment

1. Immediate -- Respond to exercises 4, 7, 8, 11, 14, 16, 17, 18, 20, and 22 (after seminar) on pages 139-142 of H & T

Task Sheet 3.1.2.1 Models for the Basic Subtraction Facts

Prerequisite:
Materials: Unifix interlocking cubes, 100-track (number line)
Problem: What kinds of subtractive situations can be modeled with Unifix materials?
Tasks:

1. Count out 14 green cubes. Join them.
   What number is modeled? ____________
   Is the model proportional or non-proportional? ____________

2. Separate 6 green cubes and return them to the box.
   How many green rods remain? ____________
   What basic subtraction fact have you modeled? ____________
   This is a model of a take-away situation.

3. Assemble the first two sections of the 100-track.
   Make a train of 6 brown cubes and place one end at line 14 on the 100-track.
   Place enough white cubes on the left of the brown train to make 14 cubes.
   How many white cubes did you use? ____________
   What basic subtraction fact have you modeled? ____________
   This is a model of an additive situation.

4. Place a train of 14 green cubes beside a train of 6 brown cubes.
   Remove enough green cubes to make them of equal length.
   How many did you remove? ____________
   What basic subtraction fact have you modeled? ____________
   This is a model of a comparative situation.

5. All three subtractive situations can be modeled with the Unifix cubes. Try enough more of the facts to ensure that you have mastered the distinction.

Task Sheet 3.1.2.2 Models for the Basic Subtraction Facts

Prerequisite:
Materials: Balance Scale
Problem: How can the balance scale be used to model subtraction facts?
Tasks:

1. Place 1 weight on the 2-peg of the left arm.
   Place 1 weight on the 9-peg of the right arm.
   Which is greater? ____________
2. Place 1 weight on the peg of the left-arm that will make the scale balance.
   On what peg did the scale come into balance? ____________
3. What basic subtraction fact was modeled? ____________
   Clear the balance scale.
4. Place 1 weight on the 9-peg of the left arm.
   Place 1 weight on the 10-peg and 1 weight on the 8-peg of the right arm.
   Which is greater? ____________
5. Place 1 weight on the peg of the left arm that will make the scale balance.
   On what peg did the scale come into balance? ____________
6. What basic subtraction fact was modeled? ____________
7. Return the weights to their storage pegs.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
6,6,3

Task Sheet 3.1.3.1 Models for the Subtraction Algorithm

Prerequisite: -
Materials: Place value cans, Soda straws, and Rubber bands
Problem: How can place value cans be used as a proportional model the subtraction algorithm?

Tasks:

1. We want to demonstrate that 45-18=27 by using place value cans. Here, we shall make use of them as a proportional representation.
2. Count out 45 straws. Use the rubber band to make 4 bundles of 10 straws each. Place the bundles in the tens can (the middle can). Place the 5 more in the ones can (the right hand can).
   You have a proportional model of what number? ____________
3. Are there enough straws in the ones can for you to remove 8? ____________
4. Remove one bundle of 10 straws from the tens can, take off the rubber band, place the straws in the ones can.
   How many bundles of 10 straws in the tens can? ____________
   How many straws in the ones can? ____________
5. Now, remove 8 straws from the ones can.
   How many are left in the ones can? ____________
6. Remove 1 bundle of 10 straws from the tens can.
   How many are left in the tens can? ____________
7. You now have a proportional model of what number? ____________
8. What number question have you answered? ____________
9. Continue to Task Sheet 3.1.2.2.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VEI
082580
6,6,4

Task Sheet 3.1.3.2 Models for the Subtraction Algorithm

Prerequisite: Completion of Task Sheet 3.1.2.1
Materials: Place value cans and soda straws
Problem: How can place value cans be used as a non-proportional model of the subtraction algorithm?

Tasks:

1. We want to demonstrate that 45-18=27, using the place value cans as non-proportional models. The straws having only red lines will represent ones. The straw having red and blue lines will represent tens.

2. Place 4 red and blue-lined straws in the tens can. Place 5 red-lined straws in the ones can.
   You have a non-proportional model of what number? ____________

3. Are there enough red-lined straws in the ones can for you to remove 8? ____________

4. Trade one 10-straw for ten 1-straws.
   Place the 1-straws in the ones can.
   How many straws in the tens can? ____________
   How many straw in the ones can? ____________

5. Now, remove 8 straws from the ones can.
   How many are left in the ones can? ____________

6. Remove 1 10-straw from the tens can.
   How many are left in the tens can? ____________
   This represents what number? ____________

7. You have a non-proportional model of what number? ____________

8. What number question have you answered? ____________

Task Sheet 3.2.2.1 Models for Subtracting Signed Numbers

Prerequisite: Task Sheet 3.2.1.1
Materials: Set of American Checkers
Problem: How can the subtraction of integers be modeled with checkers?
Tasks:

1. Make a pile of 8 red and 5 black checkers.
   What is the value of the pile? __________
2. Remove 2 red checkers.
   What operation have you represented? __________
3. What is the value of the remaining pile? __________
4. You have modeled -3 - -2 = ☐ __________
5. Now show that -3 - -2 = -3 + +2.
6. Then demonstrate these:
   -3 - +2 = -5
   +3 - -2 = +5
   +3 - +2 = +1

Part Three

This part has three units. The first two, which are concerned with the teaching of multiplication and division of whole numbers, are much like those concerned with addition and subtraction. The third is concerned with teaching the uses of common and decimal fractions and arithmetical operations with rational numbers.

These topics are given greatest emphasis in intermediate and upper grades. Arrange to observe math classes in one of these grades for about three days. This will be a part of your directed observation for these units.

Unit VII. Multiplication

A. Materials
  1. Professional Textbooks
    a. Heimer and Trueblood
    b. Ashlock
  2. Pupils textbook of your choice
  3. Manipulatives and Task Sheets
    a. Balance scale
    b. Open end abacus
    c. Graph paper

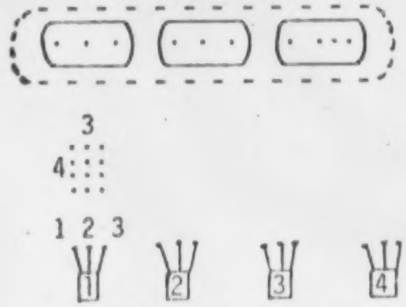
B. Study
  1. Directed observation
    a. Reflect on your observation in an intermediate or upper grade classroom. Then, refer to the comprehensive flow chart on page 161 of H & T. Which, if any, of the topics were being studied?
    b. Complete the task sheets at the end of this unit. They are intended to demonstrate to you the cumulative nature of use of two instructional devices and introduce you to another.
  2. Basic reading assignment
    a. Heimer and Trueblood - pages 146-180
    b. Ashlock - pages 136-137, 29-32, 62-65, and 94-99
  3. Comment
    a. Meanings of multiplication. H & T discuss two of three physical situations that can be translated into multiplication sentences. They are the repeated addends and cross product of sets ("pairings", when working with children). The third kind of situation involves rate pairs. Rate pairs are involved in situations like this: If there are to be three cookies for each person in the den, how many cooki are needed if all eight members are present? Notice that this situation describes a three-to-one matching between cookies and Cub Scouts. Most textbook series ignore the rate pair situations because they are so easily translated into repeated addends situations. This is unfortunate because rate pairs provide a powerful tool for problem solving.

A simple comparison among these three meanings is shown below

Repeated addends $4 \times 3 = \boxed{12}$

Pairings $4 \times 3 = \boxed{12}$

Rate Pair $4 \times 3 = \boxed{12}$



Do note that both the pairings situation and the rate pair situation can be converted to repeated addends with a bit of mental legerdemain. Unfortunately, pupils in many classes are expected to do these slight-of-mind tricks without being shown how.

The array, which is the diagram used to illustrate the pairings situation, needs some additional comment. In most of mathematics, a convention regarding arrays exists. That is, they are named as rows by columns (r x c)._ In this diagram, four rows and three columns are used to show the pairings among four things of one kind and three of another. Three rows and four columns would illustrate 3 x 4. Now, it is true that 3 x 4 = 4 x 3 and 2 x 5 = 5 x 2. However, in the world of real things, they differ. Suppose two schools have ten tug-of-war players of equal size and strength. One establishes five teams of two. The other has two team of five. The likelihood of the season ending in a tie is about 0.00.

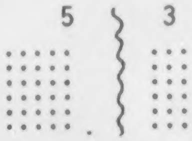
b. Basic multiplication facts. In most schools more pupils have difficulty mastering the basic multiplication facts than either the addition or subtraction facts. And then when you think a child has finally mastered all 100 of them, you find that some have been forgotten.

Several reasons have been put forth to explain the difficulties that pupils have with basic multiplication facts. Not as much time is devoted to learning them. Adults in a child's life are far more likely to ask a four year old for the answer to 5 + 4 = ☐ than an eight year old for the answer to 5 x 4 = ☐. Concrete and semi-concrete representations are not used as much or for as long. More different numbers are products than are sums. "Gaps" exist between the numbers that are products.

You will likely find more pupils needing to use a table like the one on page 171 more and longer than those who used a similar table for addition and subtraction. It is well worth spending parts of several class periods inspecting a multiplication table like the one on page 171 of H & T. Here are some of the features that should be noted:

i. The first column (and first row) are all zeroes.
ii. Each column is like a counting chart.
iii. All of the even-numbered columns have products that are even numbers.
iv. Repeating sequences of digits in ones place are found in the 2-column, 4-column, 5-column, 6-column, and 8-column.
v. The sum of the digits in any multiple of nine is 9.
vi. The table can be folded on the upper left-lower right diagonal so that 45 commutative pairs of products
vii. The square numbers (0 x 0, 1 x 1, 2 x 2, 3 x 3, etc.) are on the upper left-lower right diagonal.
viii. The product one down to the left or one up to the right of a square number is one less than the square number. The product two down and two to the left or two up and two the the right of a square number is four less than the square number.
ix. The sum of the digits of two-digit products of three is divisible by three.

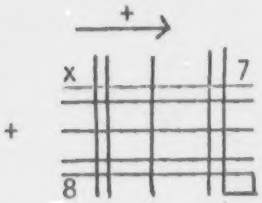
A pupil should have some way of reconstructing a basic multiplication fact if it is forgotten. This is one reason for emphasis on the squares and multiples of five. For example, if a pupil forgets the product of 6 and 8, the eight can be renamed as 6 + 2 or 5 + 3. Then, the pupil works with facts that supposedly are known 6 x 8 = 6 x (6 + 2) = (6 x 6) + (6 x 2) = 36 + 12 = 48 or 6 x 8 = 6 x (5 + 3) = (6 x 5) + (6 x 3) = 30 + 18 = 48. Of course, such a pupil might also do something like make an array or other dot drawings.



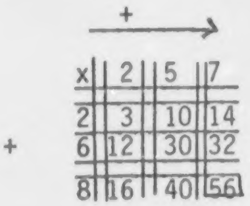
The recommendations given for helping pupils master other basic facts apply here as well: models, oral drill, spaced practice, variety, and knowledge of results. Review, especially at the beginning of each year, is important.

We especially recommend the following practice activity for its relation to the multiplication algorithm.

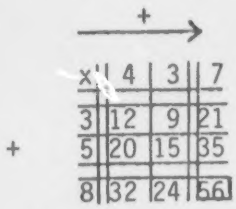
Exercise: 7 x 8 = □



Solution 1



Solution 2



Some confusion exists in how one reads a multiplication question like 4 x 3 = □. Is it "four times three equals box" or "four multiplied by three equals box"? They convey different meanings. The first implies four sets of three things. The second implies three sets of four things. We recommend that "times" be used when the questions is in horizontal form and "multipled by", reading from top down, when the question is in vertical form. The same applies to the multiplication algorith, which is almost always recorded in vertical form.

c. Multiplication algorithm. Think of all the learnings involved in finding a solution to an example like 46 x 79: basic multiplication facts (6 x 9, 6 x 7, 4 x 9, 4 x 7), renaming a product to a largest multiple of ten and ones (54 = 50 + 4), renaming a product to a largest multiple of hundreds and tens (470 = 400 + 70 or 47 tens = 4 hundreds and 7 tens and 36 tens = 3 hundreds and 6 tens), renaming a product to the largest multiple of thousands and hundreds (31 hundreds = 3 thousands and 1 hundred), higher decade addition (42 tens plus 5 tens) including bridging (28 tens plus 3 tens), basic addition facts (2 + 5 = 7, 8 + 3 = 11, 4 + - = 0, 7 + 6 = 13, 1 + 4 = 5, 5 + 1 = 6, - + 3 = 3), column addition (1 + 4 + 1), and much about place value. The opportunities for error are numerous.

A propensity exists to move too quickly from concrete and semi-concrete models of multiplication sentences to purely abstract computational forms.

The standard algorithm includes several shortcuts. If a child is to understand the algorithm, he must know how those shortcuts are made. Concrete representations are typically few in number, and understandably so. One must have 224 bottle caps or discs to illustrate the product of 14 and 16. The logistics of using proportional models at the semi-concrete level are only a little less imposing. Non-proportional models, like place value grids, are a little more manageable. Most of these models should arise in demonstrations rather than from hands-on experiences.

We recommend that before moving to the standard form of the algorithm that pupils have enough experience with longer forms, like those below, to insure some understanding of the partial products.

```
 21        42        24
 x3       x31       x38
 --       ---       ---
  3         2        32
 60        40       160
 --        60       120
 63      1200       600
         ----       ---
         1302       912
```

The exercise that we recommended in the last section provides a useful intermediate step between arrays like the one on page 158 and the intermediate algorithm above it. Notice how the sources of the partial products are clearly identified.

+ →

+ ↓

x	10	6	16
50	500	300	800
4	40	24	64
54	540	324	864

Estimating products and checking answers should be given attnetion. Estimates before computation are likely to reduce place value errors like 38. The most common check is to reverse

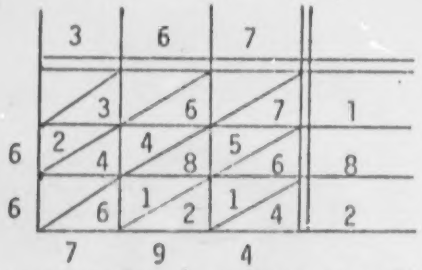
```
 38
x24
---
152
 76
---
228
```

the factors.

A low-stress algorithm is available for multiplication. A completed example is presented below:

```
 367    367     367     367     367     367
x182   x182    x182    x182    x182    x182
----   ----    ----    ----    ----    ----
               11      11      11      11
       - - -  -624-   - 624-   624 -   624 -
                      245     245     245
               ----    486-    486---  486-
               ----    ----    ----    ----
                              367     367
                              ----    ----
                                      66794
```

The lattice method, which is described on page 176 of H & T, is probably easier to teach, though the place value relationships are not so evident. The same example appears like this:



A research study reported a few years ago showed the lattice method to be as efficient as the standard algorithm and to result in greater accuracy.

Some of the other algorithms that can be used for enrichment are these:

Russian

38 x 25		or	25 x 38	
~~38~~	~~25~~		25	38
19	50		~~12~~	~~76~~
9	100		~~6~~	~~152~~
~~4~~	~~200~~		3	304
~~2~~	~~400~~		1	608
1	800			950
	950			

Egyptian

38 x 25	
~~1~~	~~25~~
2	50
4	100
~~8~~	~~200~~
~~16~~	~~400~~
32	800
	950

(notice even numbers on left) (notice sum of left column)

C. Review
1. You are probably tired by now of formulating questions related to the objectives. Go through the list of objectives on pages 146 and 147 of H & T and check those that you are pretty sure you have mastered. Then, review the sections related to those you have not checked.
2. Thumb through your pupils text. Try to match the multiplication topics with the appropriate box in the flow chart on page 161.
3. Prepare any questions on any matters you want your instructor to clarify.

D. Assignment
1. Immediate -- Respond to the exercises 3, 5, 8, 10, 13, 14, 17 (two will do), 18, 20, 22.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
7,6,1

Task Sheet 4.1.2.1 Models for the Basic Multiplication Facts

Prerequisite:
Materials: Graph paper, pencil, and straight edge
Problem: How can the basic multiplication facts be modeled with graph paper?
Tasks:

1. Draw a rectangle enclosing 1 row and 4 columns.
   How many squares in the rectangle?

2. Give your paper a quarter turn.
   How many rows?
   How many columns?
   How many squares?
   This is a model of 4 x 1 = 4.

3. Draw a model of 2 x 2 = 4.
   What is its shape?
   4 is called a square number.

4. Draw a rectangular model of 4 x 9.
   How many squares on the interior of the rectangle?

5. Use a dash line in your model of 4 x 9 to make a model of 4 x 5 and 4 x 4.
   How many squares in the 4 x 5 rectangle?
   How many squares in the 4 x 4 rectangle?
   What is 20 + 16?

6. This is a model of the distributive property of the distribution property of multiplication: 4 x (5 + 4) = (4 x 5) + (4 x 4).

7. Make models for all products of 2, 3, 5, 6, 7, 8, 9, and 10.
   Which have only one row (or column)?
   What are these numbers called? P_1_E

Curriculum and Instruction 4152 VET
Mathematics in the Elementary School Curriculum 082580
7,6,2

Task Sheet 4.1.2.2 Models of the Basic Multiplication Facts

Prerequisite:
Materials: Balance scale
Problem: How can the basic multiplication facts be modeled on the balance scale?
Tasks:

1. Place 4 weights on the 2-peg of the left arm of the balance scale. This is a model of 2+2+2+2. How is 2+2+2+2 expressed as a multiplication phrase? ______ X ______
2. Place 1 weight on the peg of the right arm that will make the scale balance. What is the value of that peg? ____________
3. You have demonstrated that 4 x 2 = ☐? ________ ____
4. Return the weights to their storage pegs.
5. Place 3 weights on the 8-peg of the left arm. What is the greatest number of weights you can put on the 10-peg of the right arm without tipping the scale to the right? ____________
6. Keep the 2-weights on the 10-peg. Place one weight on another peg of the right arm to bring the scale into balance.
7. What number is represented on the right arm? ____________
8. What basic multiplication fact have you modeled? ______ = ______
9. Return the weights to their storage pegs.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
7,6,3

Task Sheet 4.1.3.1 Models for the Multiplication Algorithm

Prerequisite: Task Sheet 4.1.1.1
Materials: Graph paper, pencil, straight edge, and scissors
Problem: How can the multiplication algorithm be modeled with graph paper?
Tasks:

1. Draw a rectangle having 26 rows and 34 columns.

2. Use your scissors to cut out the rectangle.

3. Fold the rectangle 20 spaces from the left.
   How many spaces to the right of the fold? ________
   20 + 6 = ☐ ________

4. Fold the rectangle 30 spaces from the top.
   How many spaces below the fold? ________
   30 + 4 = ☐ ________

5. Label the lower right rectangle A.
   A has what dimensions? ____ X ____
   A has how many squares in it? ________

6. Label the lower left rectangle B.
   B has what dimensions? ____ X ____
   B has how many squares in it? ________

7. Label the upper right triangle C.
   C has what dimensions? ____ X ____
   C has how many squares in it? ________

8. Label the upper left triangle D.
   D has what dimensions? ____ X ____
   D has how many squares in it? ________

9. What is the sum of the last answers in Tasks 5,6,7, 8? ________
   What is 26 x 34? ________

10. Observe the partial products in this intermediate algorithm:

$$\begin{array}{r} 34 \\ \underline{\times 26} \\ 24 \\ 180 \\ 80 \\ \underline{600} \\ 884 \end{array}$$

Then observe the partial products in this standard algorithm:

$$\begin{array}{r} 34 \\ \underline{\times 26} \\ 204 \\ \underline{68} \\ 884 \end{array}$$

11. The sum of which two rectangles equals 204? ________

12. The sum of which two rectangles equals 680? ________

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
7,6,4

Task Sheet 4.1.3.2 Models for the Multiplication Algorithm

Prerequisite:
Materials: Open-end abacus -
Problem: How can the multiplication algorithm be modeled with an open-end abacus?
Tasks:

1. Assemble the rods for the abacus.
2. Enter 1 counter on the hundreds rod, 2 counters on the tens rod, and three counters on the ones rod.
   What number is represented?
3. Enter 123 two more times.
   How is 123+123+123 expressed as a multiplication phrase?
4. How many counters on the hundreds rod?
   What is the value of these counters?
5. How many counters on the tens rod?
   What is the value of these counters?
6. How many counters on the ones rod?
7. What is 300+60+9?
8. What is the product of 3 X 123?
9. Enter 123 two more times.
   How is 123+123+123+123+123 expressed as a multiplication phrase?
10. How many counters on the ones rod?
    How many counters on the tens rod?
    What is the value of these counters?
    How many counters on the hundreds rod?
    What is the value of these counters?
11. What is 15+100+500?
12. This much like an intermediate algorithm. Let/s try a shortcut to get the standard multiplication algorithm.
13. Trade 10 ones for 1 ten.
    How many left on the ones rod?
    How many on the tens rod?
14. Trade 10 tens for 1 hundred.
    How many left on the tens rod?
    What is its value?
15. How many on the hundreds rod?
    What is their value?
16. What is 5+10+600?
17. 5 X 123 = ☐?

Unit VIII. Division

A. Materials
  1. Professional textbooks
    a. Heimer and Trueblood
    b. Ashlock
  2. Pupils' textbook at an intermediate or upper grade level
  3. Manipulatives and Task Sheets
    a. Arithmestix
    b. Discs

B. Study
  1. Directed observation

a. Reflect on your observation in preparation for this part. Were pupils doing any study of division? If so, were they studying the basic facts? The "uneven" division facts? Were they studying an algorithm? Was it the same one you were taught? The same one you now use? If not, could you identify any concepts or principles related to division that were being studied? Multiplication sentences with unknown factors? Statements like 14 = (3 x 4) + 2 with any number as an unknown? Fractions as representing parts of something?

b. Complete the task sheets, which are found at the end of this unit. Only two are asked of you this time. One is a sample of the proportional representations that can be used. The other is a non-proportional device.

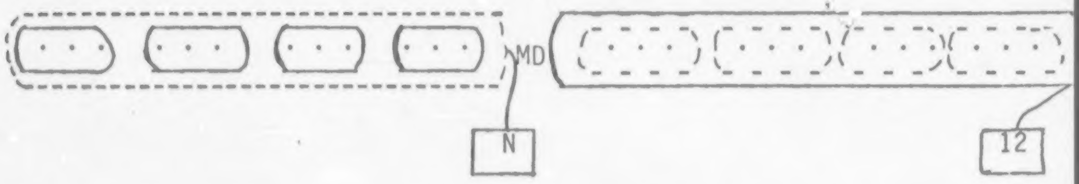
  2. Basic reading assignment
    a. Ashlock -- pages 137, 33-36, 65-68, and 100-106
    b. Heimer and Trueblood -- pages 188-214

  3. Comment

a. Meanings of division. In Unit VI, three physical situations that can be translated to multiplication sentences were identified. They were referred to as repeated addends, pairings, and rate pairs. Two division situations are related to each of the multiplicative situations. The same diagrams will be used in this illustration:

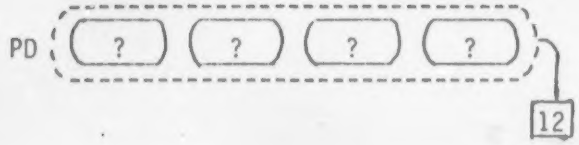
MULTIPLICATION RELATED DIVISION

Repeated Addends



The number in the superset is unknown. The number of sets is unknown.

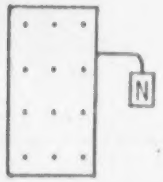
$4 \times 3 = \square$ $\square \times 3 = 12,\ 12 \div 3 = \square$



The number in each set is unknown.

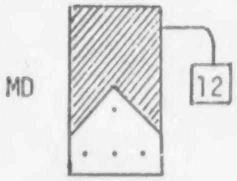
$4 \times \square = 12,\ 12 \div 4 = \square$

Pairings



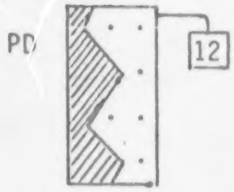
The number of pairings is unknown.

$4 \times 3 = \square$



The number of rows is unknown.

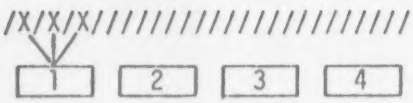
$\square \times 3 = 12,\ 12 \div 3 = \square$



The number of the columns is unknown.

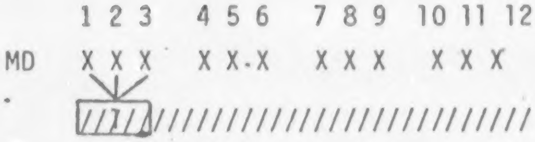
$4 \times \square = 12,\ 12 \div 4 = \square$

Rate Pairs



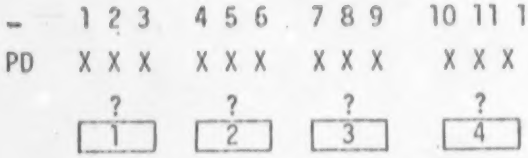
The total number of the greater sets is unknown.

$4 \times 3 = \square;\ 3/1 = \square/12$


MD
1 2 3
4 5 6
7 8 9
10 11 12
X X X
X X X
X X X
X X X
///1///////////////////////////////

The number of sets of one is unknown.

$\square \times 3 = 12,\ 12 \div 3 = \square;\ 3/1 = 12/\square$


1 2 3 4 5 6 7 8 9 10 11 12
PD
X X X X X X X X X X X X
?
?
?
?
1
2
3
4

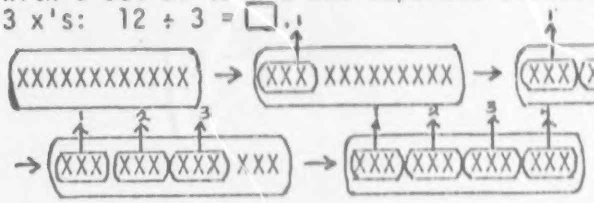
The number in each of the greater sets is unknown.

$4 \times \square = 12$, $12 \div 4 = \square$; $\square/1 = 12/4$

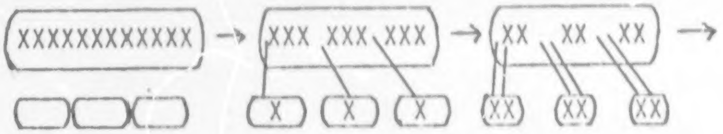
All of those divisive situations marked MD are referred to as measurement situations. Those marked PD are referred to as partition situations. Fortunately, not all of these learnings have to be developed. Almost all study of division situations relates to the repeated addends situations of multiplication. A little attention is given to rate pairs, especially in the study of measurement. Virtually no attention is given to pairing. However the "tiger bites" used above are found in a few textbook series.

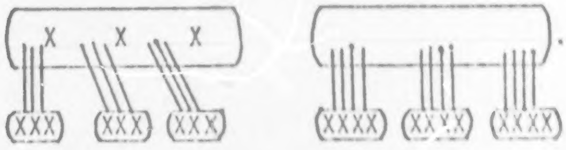
Most mathematics educators look on measurement situations as being more basic than partitive situations. This is probably because of the pedagogical ease of relating division, as serial subtraction, to multiplication, as repeated addends. By the way, the terms "measurement division" and "partitive division" are rarely used with pupils. They are teacher-talk. However we do make frequent use of "the number of sets" when referring to measurement situations and "the number in each set" when referring to partitive situations.

Let's take one more look at the difference. We begin with a set of 12 x's and separate successive sets of 3 x's: $12 \div 3 = \square$.


XXXXXXXXXXXX
XXX XXXXXXXXX
XXX XXX XXXXXX
XXX XXX XXX XXX
XXX XXX XXX XXX
1
2
3
4

Aha!! There are four sets of three in a set of twelve. $12 \div 3 = 4$. Then, we turn to a partitive situation. We have a set of 12 x's that we want to separate into three sets having the same number in each.


XXXXXXXXXXXX
XXX XXX XXX
X
X
X
XX XX XX
XX
XX
XX


X X X
XXX
XXX
XXX
XXXX
XXXX
XXXX

Aha!! There are four in each of three sets from a set of twelve. $12 \div 3 = 4$.

b, Basic division facts. Proposals have been made to require pupils to master neither the basic subtraction facts nor the basic division facts. Both have as their rationale that these facts have equivalent expressions in additive and multiplicative sentences. A basic subtraction question, like $12 - 4 = \square$, is equivalent to open addition sentence, like $\square + 4 = 12$. A basic division question, like $12 \div 4 = \square$, is equivalent to an open multiplication sentence, like $\square \times 4 = 12$.

The argument is more persuasive for division than for subtraction. We have observed the problems that primary grade children have with open sentences. However, many adults do not make much use of division facts in using the standard division algorithm. In finding the first estimated partial quotient when solving $43\overline{)1548}$, many adults will think "Now let's see. What is the largest multiple of four that is less than 15? Oh yes, four times three equals twelve. The difference is less than four. I'll put a three over the four in the dividend."

Despite the proposal, the basic division facts are taught and taught for mastery and are very likely to be a part of the curriculum for many years to come. In fact, our guess is that if curricula ever change to obviate the requirement of pupils trying to master the division facts the rationale will be another. That rationale will likely be the availability of calculators.

The kind of exercise shown near the bottom of page 193 in H & T provides a nice introduction to shifting from horizontal notation to vertical notation. The Greenwood algorithm provides a convenient way of recording the shift. A suggested sequence is shown below:

i. $$\begin{array}{rr} 6 & \\ \underline{-2} & 1 \\ 4 & \\ \underline{-2} & 1 \\ 2 & \\ \underline{-2} & \underline{1} \\ 0 & 3 \end{array}$$

ii. $$\begin{array}{r|r} 2\overline{)6} & \\ \underline{-2} & 1 \\ 4 & \\ \underline{-2} & 1 \\ 2 & \\ \underline{-2} & \underline{1} \\ 0 & 3 \end{array}$$

iii. $$\begin{array}{r|r} 3 & \\ 2\overline{)6} & \\ \underline{-2} & 1 \\ 4 & \\ \underline{-2} & \underline{2} \\ 0 & 3 \end{array}$$

iv. $$\begin{array}{r|r} 3 & \\ 2\overline{)6} & \\ \underline{-4} & 2 \\ 2 & \\ \underline{-2} & \underline{1} \\ 0 & 3 \end{array}$$

v. $$\begin{array}{r|r} 3 & \\ 2\overline{)6} & \\ \underline{-6} & \underline{3} \\ 0 & 3 \end{array}$$

The use of facts that pupils presumably know when introducing a computational algorithm permits the teacher to attend to the dictum, "One difficulty at a time." The new algorithm also provides pupils with an opportunity to study the so-called "uneven facts," like $25 \div 4 = \boxed{6R1}$ or $85 \div 9 = \boxed{9R4}$. They might appear in various forms at this stage.

i. $$\begin{array}{r|r} 6R1 & \\ 4\overline{)25} & \\ \underline{-12} & 3 \\ 13 & \\ \underline{-8} & 2 \\ 5 & \\ \underline{-4} & \underline{1} \\ 1 & 6 \end{array}$$

ii. $$\begin{array}{r|r} 6R1 & \\ 4\overline{)25} & \\ \underline{-4} & 1 \\ 21 & \\ \underline{-20} & \underline{5} \\ 1 & 6 \end{array}$$

iii. $$\begin{array}{r|r} 9R4 & \\ 9\overline{)85} & \\ \underline{-45} & 5 \\ 40 & \\ \underline{-18} & 2 \text{ or} \\ 22 & \\ \underline{-18} & \underline{2} \\ 4 & 9 \end{array}$$

iv. $$\begin{array}{r|r} 9R4 & \\ 9\overline{)85} & \\ \underline{-81} & \underline{9} \\ 4 & 9 \end{array}$$

Division by zero can become a sticky point with some thoughtful pupils. They are reluctant to accept your statement that division by zero is undefined. The best way to deal with the question if it ever arises is to relate division to open multiplication sentences in an inductive sequence something like this: $12 \div 6 = 2$ because $2 \times 6 = 12$, $12 \div 4 = 3$ because $3 \times 4 = 12$, $12 \div 2 = 6$ because $6 \times 2 = 12$, $12 \div 0 = \square$, now just what number times zero will equal twelve? Since we can find no such number we have to say that division by zero is not defined. Then, some wiseacre may ask $0 \div 0 = 0$. How will you respond? (HINT: Look up the term indeterminate in your dictionary and consider whether $0 \div 0 \overset{?}{=} 1$, $0 \div 0 \overset{?}{=} 2$, $0 \div 0 \overset{?}{=} 587$.)

Most curricula attempt to have pupils learn the basic division facts if not at the same time they are learning the basic multiplication facts then near the same time. A great number of practice exercises are designed to verify the relationship between them. Do not be surprised if you find a few pupils having to use their multiplication tables if they are to be successful in the use of a division algorithm.

The recommendations that were given for mastering other kinds of basic number facts stand for division. One difference is the so-call "uneven facts." They should not be learned as facts. There are too many (360) of them. Rather, they should be related to known facts. In fact, study of them provides readiness for an important step in conventional division algorithms. That step is finding the largest multiple of some number (the divisor) that is less than another number (a partial dividend).

c. Division algorithms. Survey tests given to students of middle school age, typically show more errors in the use of the division algorithms than any other. Inspection of the solution to an exercise like $27\overline{)4378}$ will give you a clue as to why. All four operations are used. A sound understanding of place value is essential. The steps are several and decisions (too large, too small) have to be made.

Few adults use a division algorithm in exactly the same way they were taught. You can expect some of your students to improvise.

The standard division algorithm that most of us learned as youngsters, like other standard algorithms, makes a near minimal record of the thought used in finding a solution. You should have observed that H & T recommend more detailed records in the stages described on pages 196-198. This is very interesting in that the form is very closely akin to the Greenwood algorithm. Let us say a few words about the Greenwood algorithm.

The Greenwood algorithm was identified with the so-called new math of the mid-sixties and early-seventies. The algorithm appears in Isaac Greenwood's book, Arithmetic, Vulgar and Decimal. The book was the first English-language mathematical text that was both written and published in North America. It was published in 1729.

The algorithm was presented in the model textbooks that were written for the School Mathematics Study Group in the early-sixties. It was later incorporated into many commercial textbook series. Some presented only the Greenwood algorithm. Others used it for introduction and then changed to the standard algorithm as students developed facility in its use. To say that the Greenwood algorithm was grossly misunderstood and taught poorly is an understatement. Inspect the one exercise solved both ways below. (The parentheses are used to show how the partial quotients were secured.)

Standard Algorithm

```
     244R8
24 /5864
   -48      (5T ÷ 2t ≈ 2h)
    106
    -96     (10h ÷ 2t ≈ 5t → 4t)
     104
     -96    (10t ÷ 2t ≈ 5 → 4)
       8
```

Greenwood Algorithm

```
     244R8
24 /5864 |
   -2400 |  100 (3T ÷ 3t = 100)
    3464 |
   -2400 |  100 (3T ÷ 3t = 100)
    1064 |
    -720 |   30 (9h ÷ 3t = 30)
     344 |
    -240 |   10 (3h ÷ 3t = 10)
     104 |
     -72 |    3 (9t ÷ 3t = 3)
      32 |
     -24 |    1 (3t ÷ 3t = 1)
       8 |  244
```

The Greenwood algorithm is shown in "pure" form. Pure form requires that the divisor always be rounded up ($24 \approx 30 = 3$ tens) and that the dividend or partial dividend always be rounded down to some largest multiple of the rounded divisor ($5864 \approx 3$ thousand, $3464 \approx 3$ thousand, $1064 \approx 9$ hundreds, $344 \approx 3$ hundreds, $104 \approx 9$ tens, $32 \approx 3$ tens).

The Greenwood algorithm was said to have these advantages: i. All dividends and partial dividends (5864, 3464, etc.) appear as "whole" numbers. ii. All partial quotients appear as "whole" numbers (100, 100, 30, etc.). iii. No overestimates of partial quotients will occur. Underestimates may occur (as here $100 + 100 + 30 + 10 + 3 + 1$) but it makes no difference. Some disadvantages are obvious: i. The rounding procedures are markedly different from others that are used in arithmetic. ii. More space is used. iii. The pure form may not utilize information that is readily available to some users ($2 \times 24 < 58 < 3 \times 24$). Proponents countered the last disadvantage by encouraging users to make more efficient estimates of quotients until they were using a close approximation of the standard algorithm, as below:

```
24 /5864
   -4800   200
    1064
    -960    40
     104
     -96     4
       8   244
```

From this has come the pyramid algorithm, which is shown on page 198 as Form B in H & T and on page 105 of Ashlock.

The writers of this syllabus recommend the Greenwood algorithm for initial study of long division. Less capable learners should continue its use. More capable learners may well change to the standard algorithm since it has greater social acceptance.

Regardless of which algorithm is used, pupils should be very proficient in rounding numbers and dividing rounded numbers before undertaking study of long division. For example, 2468 rounds to one or two significant digits as 2000 or 2500; 47 rounds to 50; $200\cancel{0} \div 50 = 20h \div 5t = 4t$.

Shortcuts are encouraged, like $\frac{200\cancel{0}}{5\cancel{0}} = 40 = 4$ in tens place.

Reference has been made to long division. Short division is a shortcut used with division questions like $8\overline{)2768}$, where the divisor is a one digit numeral. A solution has this appearance. $\begin{array}{r} 3\;4\;6 \\ 8\overline{)2\;7^3 6^4 8} \end{array}$. If taught at all, short division should come <u>after</u> pupils are proficient with divisors having two or more digits. Otherwise, great numbers of systematic errors occur.

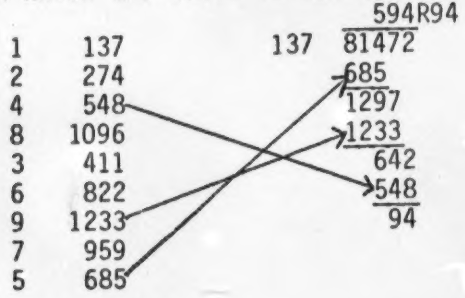
The question often arises as to how remainders should be treated. For exercises, the answer is "however the teacher wants." Your options are that remainders i) be expressed as whole numbers, as was done above (mathematical purist object to this way, but the practice is a part of our culture.) ii) expressed as a fraction iii) dropped or iv) that the quotient be rounded up to the next higher whole number. For applications, the answer is different. The problem dictates that manner in which remainders will be treated. Consider these simple situations.

1. We have 29 pupils in our class. Each car can carry 6 pupils, no more, no less. How many cars are needed? (Rounded up.)
2. We have 29 pupils in our class. Each car can carry 6 pupils, no more, no less. How many cars are needed and how many pupils can't go? (Retain remainder as a whole number.)
3. I have 36 photographs. My album will permit me to put 8 on a page. How many full pages can I fill? (Ignore the remainder.)
4. I have 36 inches of string. I have been instructed to cut it into 8 equal pieces. How long should each piece be? (Express as a fraction.)

As you have come to expect, alternative algorithms are available. The low-stress algorithm looks like this:

$$\begin{array}{l|l}
137 & \overline{31472} \\
13 & 704 \\
555 & \underline{685} \\
\boxed{685} & 1297 \\
26 & \\
973 & 1233 \\
\boxed{1233} & 642 \\
12 & 532 \\
428 & \underline{548} \\
\boxed{548} & 94
\end{array}$$

The low stress algorithm is rather imposing in appearance. Though no research evidence is known in which comparisons with the Greenwood algorithm have been made, we suggest it as an alternative to the standard algorithm. Also, many pupils are benefited by being permitted to build tables as below (note the order in which the table is built):



If the less-capable learners "just can't get it," teach them to estimate products with rounded numbers, if possible, and to use a calculator with care.

The algorithms for enrichment are, for the most part, designed for special situations. Some are these:

Complement method. This can be used for divisors that are slightly under 100.

$$98\overline{)76814} \rightarrow 100-2\overset{\displaystyle 783}{\overline{)76814}} \quad \begin{array}{r} +14 \\ \hline 821 \\ +16 \\ \hline 374 \\ +6 \\ \hline 80 \end{array}$$

$$96\overline{)44448} \rightarrow 100-4\overset{\displaystyle \overset{3}{462}}{\overline{)44448}} \quad \begin{array}{r} +16 \\ \hline 604 \\ +24 \\ \hline 288 \\ +8 \\ \hline 96 \end{array}$$

Austrian method. This is used in much of South America. Notice that it makes an even briefer record of thought than out standard algorithm.

$$\begin{array}{r} 27R10 \\ 73\overline{)1981} \\ 521 \\ 10 \end{array} \qquad \begin{array}{r} 398R2 \\ 241\overline{)95920} \\ 2362 \\ 1930 \\ 2 \end{array}$$

Adanda method. This has enjoyed some use in other cultures.

```
      27R10                 3 8
  73/1981                 ~~4~~9~~9~~R2
    -14               241/95920
 -   58                   -6
    - 6                   359
    521                   -12
    -49                   239
     31                   -  3
    -21                   2362
     10                   -18
                           562
                           -36
                           202
                           -  9
                           1930
                           -16
                            330
                            -32
                             10
                            - 8
                              2
```

The most common procedure in checking a quotient is to have pupils multiply the quotient times the divisor and add the remainder to the product. The sum should equal the dividend. On occasion, especially if additional practice is needed, one can have pupils divide by the quotient to obtain, hopefully, the divisor. Still another that can be used once in a while is this:

```
        398
  241/95920
      723xx
       2362
      2169x
       1930
       1928
          2
      95920
```

C. Review

1. Make sure that you can perform long division yourself. You might be amazed that a few teachers have forgotten.
2. Check your capabilities against the objectives on pages 188-189 of H & T.
3. Thumb through your pupils text and identify this content, if there: readiness for division, developmental activities, the final form of the algorithm(s).
4. Prepare any questions on matters that you want your instructor to clarify.

D. Assignment

1. Immediate -- Respond to the exercises 1, 5, 6, 7, 11, 12, 14, 16, 18, 19 (recall the discussion of the number line in Unit V)on pages 216-217 of H & T.

Curriculum and Instruction 4152
Mathematics in the Elementary School Curriculum

VET
082580
8,10,1

Task Sheet 5.1.2.1 Models of the Basic Division Facts

Prerequisite:
Materials: Plastic discs
Problem: How can discs be used to model the division facts?
Tasks:

1. Count our 24 discs.
   Place them in a pile.

2. Remove a set of four discs.
   Remove another set of four discs.
   Remove as many sets of four discs as possible.
   How many sets of four discs did you remove?

3. Removing successive sets having the same number of elements is called serial subtraction.
   Division can be defined in terms of serial subtraction.
   Divisive situations of this kind are referred to as measurement division.

4. You have shown that 24 ÷ 4 = ☐?

5. 24 ÷ 4 = ☐ can also be used to represent another kind of divisive situation. It is called portitive division.

6. Make a pile of your 24 discs. Inspect the diagram below.

   How many cells in the diagram?

7. Place 1 disc in each cell.
   Repeat until all discs have been placed on the diagram.
   How many discs in each cell?

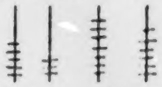
8. Partive division answers the question, "How many in each set?"
   Measurement division answers the question, "How many sets?"

9. Try representing at least two more division facts with the discs.

10. Return the discs to their container.

Task Sheet 5.1.3.1 Models of the Standard Division Algorithm

Prerequisite:
Materials: Arithmestix, scratch paper, and pencil
Problem: How can arithmestix be used to model the division algorithm?
Tasks:

1. Represent 5376 on the arithmestix:  You could find the number of sets of 8 in 5376 by serial subtraction. That would not be very efficient.
2. Remove the unused stix. Separate the thousands stick from the other three. How many sets of 8 in a set of 5? ____________
3. Since 5 represents 5 thousands or 50 hundreds, you will need to think of 5 thousands as 50 hundreds.
4. Separate the hundreds stick from the other two and join it to the thousands stick. What number is now represented? ____________
Remember, you have represented 53 hundreds.
5. Remove as many full sets of 8 as you can from 53.
How many sets did you remove? ____________
How many hundreds did you remove? ____________
6. The 6 represents the number of sets of 8 in the number of hundreds in the dividend. Write the 6 in hundreds place in the quotient.
8/5376
48
7. How many hundreds were not removed? ____________
8. You will need to think of the 5 hundreds as 50 tens.
Separate the tens stick and join it to the other two.
What number is now represented? ____________
Remember, you have represented 57 tens.
9. Remove as many full sets as you can from 57.
How many sets did you remove? ____________
How many tens did you remove? ____________
10. The 7 represents the number of sets of 8 in the number of tens in the partial dividend.
Write 7 in tens place in the quotient.
6
8/5376
48
57
56
11. How many tens were not removed? ____________
12. You will need to think of the 1 ten as 10 ones. Join the ones stick to the other three. What number is now represented? ____________
Remember, you have represented 16 ones.
13. Remove as many full sets of 8 as you can from 16.
How many sets did you remove? ____________
How many ones did you remove? ____________
14. The 2 represents the number of sets of 8 in the number of ones in the partial dividend. Write 2 in ones place in the quotient.
67
8/5376
48
57
56
16
16

Task Sheet 5.1.3.1 Models of the Standard Division Algorithm (Page 2)

15. Let's take another view of what was done above.

$$5376 = 5300 + 76$$
$$53 \div 8 = 6R5$$
$$576 = 570 + 6$$
$$57 \div 8 = 7R1$$
$$16 \div 8 = 2$$

16. We could also view it in another way.

$$\begin{array}{rr} & 600 + \ 70 + \ 2 \\ 8\overline{)5376} = & 8\overline{)4800 + 560 + 16} \end{array}$$

17. This use is a little tricky. Be sure to try at least two more exercises.

18. Reassemble the arithmestix and return them to their container.

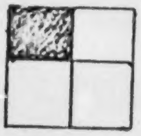
IX. Fractions

A. Materials
  1. Professional textbooks
    a. Heimer and Trueblood
    b. Ashlock
  2. Manipulatives and Task sheets
    a. Cuisennaire rods
    b. Fraction kit
    c. Egg cartons

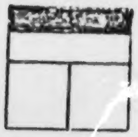
B. Study
  1. Directed observation
    a. Complete the task sheets. They can be found at the end of the unit.
  2. Basic reading assignment
    a. Ashlock -- pages 137-139, 37-51, 69-81, 106-119.
    b. Heimer and Trueblood -- pages 220-249.
  3. Comment

a. Meanings of fractions. The first sentence on page 223 is accurate, but it is also incomplete. Fractions (common fractions) are used as names for rational numbers. Rational numbers can also be named as decimal fractions -- 0.666. . . and 0.8 -- and as ordered pairs -- (2,3) and (4,5). Fractions are also used to represent situations in the real world. They can represent one or more parts of one or more units as shown in Sample Activity 1 on page 242 of H & T. They can represent one or more parts of one or more sets as shown in Sample Activity 2 on page 243. In addition a fraction is used as a ratio to compare two sets of like things and as a rate to compare two sets of unlike things. Moreover, a fraction can be used to indicate division, as 8 ÷ 4 = 8/4 = 2.

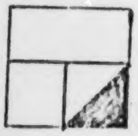
Most curricula introduce fractions as representing one or more parts of one unit: a circle divided into congruent parts, a cookie cut into parts having the same size and shape. This introduction is quickly followed by fractions as representing one or more parts of one set. Few children have difficulty encompassing both conceptions. Initial work with models of fractions usually involves identical parts. Do keep in mind that the parts do not have to be identical for them to be represented with a fraction. (The unit of reference may change.)



One fourth of the square is shaded.



One fourth of the square is shaded.



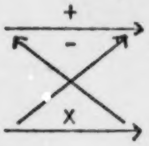
One fourth of the parts of the square is shaded.

We caution you to provide many concrete and semi-concrete experiences in establishing the equivalency of fractions before relying solely on computation. The computation is so easy for most pupils that they can do the computations without possession of the concepts. The same applies to ordering rational numbers that are expressed as fractions.

b. Computation. Rather than comment on each of the operations with fractions, we shall make some rather random remarks.

Many curricula use a slight variation on the sequence found in the flowchart on page 229 of H & T. Between case II and case III special attention is given to fractions in which one denominator is a multiple of the other, as in 3/4 + 5/8 and 2/3 + 5/12.

This sequence is interesting from the standpoint of curriculum construction. The procedures in case III are generalizable to all additions and subtractions of fractions. Curriculum builders usually try to draw on the most generalizable information available. Yet, here we find one or two special procedures, which are essentially shortcuts, being taught before the most generalizable procedure. Though consistent with the mathematical definition for addition of rational number $(\frac{a}{b} + \frac{c}{d} = \frac{(d \times a)+(b \times c)}{2 \times 4}) =$ the more generalizable procedure is difficult to rationalize using manipulative materials. Curriculum builders have decided in favor of pedagogical ease over mathematical generalizability. The recommendation has appeared recently in the literature on remedial teaching that slow learners use only the more generalizable procedure with this diagram as their mnemonic device:



. Thus, $\frac{1}{2} + \frac{3}{4}$ would be solved as $\frac{1}{2} + \frac{3}{4} = \frac{(4 \times 1) + (3 \times 2)}{2 \times 4} = \frac{4 + 6}{8} = \frac{10}{8}$.

The answer to the last example raises another issue: Should answers be reduced to lowest terms? They are certainly easier to grade of all correct answers are the same. Requiring pupils to reduce answers takes advantage of an opportunity for them to practice a skill. We do give a lot of attention standard names for numbers. So for most exercises, the answer is "usually yes." The answer is much different if the numbers represent measures. Accuracy and precision of measurement are taught at junior high school levels, if at all. In most practical situations, one would find that a measure of 1/2 was in all likelihood secured with the same instrument as the $\frac{3}{4}$. Thus, $\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$ would be quite an acceptable answer.

We encourage that pupils be required to use this form for their adding and subtracting common fractions:

$\frac{1}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{2 + 3}{4} = \frac{5}{4}$ and $\frac{7}{8} - \frac{3}{4} = \frac{7}{8} - \frac{6}{8} = \frac{7 - 6}{8} = \frac{1}{8}$.

It emphasizes that one operates on only the numerators and that the denominator merely indicates the size of the parts.

The flowchart on page 236 of H & T again shows the conflict between special cases and the most generalizable algorithm. Here, we would eliminate the special cases.

Two alternatives are available for the division algorithm that is shown on pages 238-239 of H & T. One is called the common denominator method. The solution to 5/8 ÷ 2/3 = □ looks like this: $\frac{5}{8} \div \frac{2}{3} = \frac{3 \times 5}{3 \times 8} \div \frac{8 \times 2}{8 \times 3} = \frac{15}{24} \div \frac{16}{24} = \frac{15 \div 16}{1} = \frac{15}{16}$.

It is somewhat easier to develop with models and does parallel the computational form for multiplication. The other is called the whole number method. The solution to 5/8 ÷ 2/3 = □ looks like this: $\frac{5}{8} \div \frac{2}{3} = (\frac{24}{1} \times \frac{5}{8}) \div (\frac{24}{1} \times \frac{2}{3}) = 15 \div 16 = \frac{15}{16}$. The only available research evidence supports the teaching of the invert-and-multiply method.

The last example brings up another question: Is "cancellation" taught? If so, when? It is taught as a shortcut after pupils have mastered the computational algorithm for multiplying fractions.

The next decade or two may see marked changes in how we develop understandings of and computations with fractions and decimals. The use of the metric units of measurement may well be the impetus for change. The research on sequences used in presenting them shows no differences in achievement between fractions-before-decimals and decimals-before-fractions.

C. Review

1. Check off those objectives on pages 220-221 that you have accomplished. Bone up on the others.
2. Prepare any questions on matters that you want the instructor to clarify.

D. Assignment

1. Immediate -- Respond to exercises 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, on pages 249-251 in H & T.

Part Four

Units X and XI have to do with the topics from geometry and measurement that are commonly included in the elmentary school mathematics curriculum. In contrast with topics from arithmetic, which are commonly rather highly structured and fairly consistent in order and level of presentation from textbook series to textbook series, topics from geometry and measurement may well not have that structure or consistency.

Though not all topics in measurement have an analogue in geometry, many do. Well structured curricula take advantage of this relationship.

It is a rather common, but unfortunate, practice for teachers to delete the study of geometry and measurement. Several reasons are given. Some teachers feel less comfortable in teaching geometry and measurement than in teaching arithmetic. Many schools place greater emphasis on computational proficiency. Yet, many children who are not very successful with arithmetic can be successful with geometry and measurement. Good teaching of these topics, especially measurement, requires lots of hands-on experiences.

If your background in geometry and measurement is a bit shaky, you may well want to study in these two yearbooks of the National Council of Teachers of mathematics.

NCTM. Geometry in the Mathmatics Curriculum. (Kenneth B. Henderson, Ed.) Thirty-Sixth Yearbook of the National Council of Teachers of Mathematics. Reston, Virginia: The Council, 1973. Especially chapters I, II, and III.

NCTM. Measurement in School Mathematics. (Robert E. Reys, Ed.) 1976 Yearbook of the National Council of Teachers of Mathematics. Reston, Virginia: The Council, 1976. Especially chapters I-VII, X, and XII.

X. Geometry

A. Materials
  1. Professional textbook -- Heimer and Trueblood
  2. Scope and sequence chart (found in most teacher's editions of math tests)
  3. Manipulatives
    a. Geo-stix
    b. Geoboard
    c. Solids

B. Study
  1. Directed observation
    a. Complete the task sheets, which are found at the end of this unit.
    b. Locate the geometry category in the scope and sequence chart. Do a quick self-check to determine whether you feel you have adequate information about each topic to teach it. Keep your self-check in mind as you complete the basic reading assignment.
  2. Basic reading assignment
    a. Heimer and Trueblood -- pages 256-278.
    b. Other professional sources as necessary.
  3. Comment

In the first paragraph on page 256, H & T use the expression "informal geometry." Some authorities use "intuitive geometry" in the same sense. The terms are used to distinguish between instruction in elementary schools, which is almost all based on sense experiences, and instruction in secondary schools, which emphasizes formal proofs based on certain axioms and theorems.

At least five reasons exist for the study of geometry i.) The study of geometry helps children become cognizant of size and shape in their environments. ii.) The facts, concepts, and principles learned in elementary schools provides a foundation for the systematic study of geometries and some other mathematical areas at more advanced levels. iii.) Ideas from geometry are used in other parts of elementary school curricula, including other parts of the mathematics curriculum. iv.) Though limited in number, some applications of geometric ideas can be made in everyday life. v.) The learning of geometric ideas is an excellent vehicle for learning some of the ways that mathematicians create mathematics.

Foreign observers frequently comment on the abstract nature of instruction in geometry in American elementary schools. Their comments are based on two kinds of observations: the great dependence on paper and pencil exercises in the development of most topics and the sequences of topics, which very generally is one-dimensional before two-dimensional before three dimensional. Proposals to begin the study of geometry with three-dimensional objects, like chunks of plastecine or play-dough have met with little success.

We urge you to exercise extreme care in selecting models of geometric ideas, especially in their introduction. For example, a straight piece of wire would be a better model of a line segment than a quarter-inch dowel rod because of the difference in diameter or a soda straw because of the distracting feature. A piece of paper (flat or course) would be a better model of a part-plane than a pad of paper because of the lesser thickness. After the introduction and a fair grasp of the concept being studied, other more convenient models can be used.

The matter of precision in terminology is debated. We encourage you to use precise terminology as long as you communicate. We encourage you also to permit your pupils to be less precise as long as important learnings are not at stake. For example, we would encourage you to refer to 4 x 8 cutout from graph paper as a rectangular region but permit pupils to refer to it as a rectangle if the distinction is not important in the context in which it is used.

Certain concepts in geometry require an appreciation of infinitely large. Examples are line, ray, angle, plane, and space. Other concepts require an appreciation of infinitely small. Examples are point and "width" of a line. Conceptions of infinitely large and infinitely small are characteristics of children in what Piaget refers to as the formal operations stage. Most children enter this stage at about eleven years of age. Before that age, their conceptions are based on physical representations. A child may be able to parrot a statement like, "A line segment contains an infinite number of points" without having a very good idea of how many that is. Thus, continued reference to models is essential.

The geometric constructions included in H & T are the ones most commonly found in elementary school mathematics curricula. The reason for delaying the use of compasses is debatable. A special workbook series, Geometry for the Primary Grades (now published as Fun with Geometry), was designed for use by children in grades II and up. A better reason for delay is the importance attached to constructions. Note that most constructions are presented without reasons why they work as they do.

C. Review
1. Do another quick self-check of your information about the topics in the scope and sequence chart.
2. Confirm for yourself that you accomplished the objectives on page 256 of H & T.
3. Prepare any questions you have.

D. Assignment
1. Respond to exercises 2-11 on pages 278-279 of H & T.

Unit XI. Measurement

A. Materials
  1. Professional Textbooks
     a. Heimer and Trueblood
     b. Ashlock
  2. Manipulatives
     a. Cuisennaire rods
     b. Metric rule
     c. Graph paper
     d. Trundle wheel

B. Study
  1. Directed observation
     a. Think back on your experience as a child in elementary schools. Answer these questions for yourself:
        1) Were you ever given occasion to compare one object with another for size without measuring tools?
        2) Did you ever use non-standard units measure, like a thumbs-width, for length or a piece of paper for surface?
        3) Did you measure anything outside the pages of a textbook?
        4) Did you learn any formulas by rote?
        5) Did you ever use metric units?
     b. Now, think of your life as an adult.
        1) Do you ever compare objects for size without measuring?
        2) Do you ever use non-standard units?
        3) What are some things, if any, you measured in the last month?
        4) Can you remember the formula for say the perimeter for a rectangle, the circumference of a circle, the area of a trapezoid, and the volume of a cone?
        5) Do you ever use metric units?
  2. Basic reading assignment.
     a. Heimer and Trueblood -- pages 284-311.
     b. Ashlock -- pages 52, 81, and 120.
  3. Comment

We shall begin with some definitions since several terms related to measurement are used rather loosely. All are taken from the 1976 yearbook of the National Council of Teachers of Mathematics.

i.) Measureable attribute -- a characteristic that can be quantified by comparing it to some unit.

ii.) Measuring -- the process of comparing an attribute of a physical object to some unit selected to quantify that attribute.

iii.) Measurement -- a) the result of measuring and b) a number and a unit of measure, e.g., 12mm or 20kg.

iv.) Measure -- a) the result of measuring and b) a number alone that can be derived only when the unit of measure is understood or is available for examination.

The attributes most commonly studied in elementary schools are these: length, area, volume, capacity (which is defined in terms of space occupied, or volume), angularity, mass (or more commonly, weight), temperature, and time. Of these attributes only the last two cannot be measured directly. They require indirect measurement.

In fact, most of the study that is done in elementary schools regarding time and temperature is concerned only with the dial reading. That is not intended as a desparaging remark. Dial reading is important.

A rather common sequence should be used in teaching various kinds of direct measurement. In skeleton form, the sequence proceeds in this fashion:

i.) Making crude comparisons of selected attributes of two objects
ii.) Counting non-standard units
  a) Several units applied to the attribute
  b) One unit applied several times
iii.) Counting standard units
  a) Several units
  b) One unit
iv.) Using the tool(s) of measurement
v.) Using formulas, where appropriate

Children need enough hands-on experience with measuring that they can make reasonably accurate estimates of measurements. This means that children should have many experiences in measuring various attributes of objects in the world around them. Since measuring tools are rarely available in sufficient numbers for these experiences to be all-class activities, plan to incorporate them into group and individual learning center activities. (Just what is the mass of a basketball, anyway?)

Stemming from pupils' work with estimating and measuring objects in their environments are reference measures. These should be developed concurrently with the study of various kinds of measurements. Some examples are these: The nail of my finger is about 1 cm. wide. My hand is about 1 dcm. wide. My hip pocket is about 1 m. above the ground. My math book has a mass of 1 kg.

Some teachers look on the units of the metric (SI for Systeme International d'Unités) system with trepidation. No good reason other than unfamiliarity exists for this reaction. The metric units are presented in the same fashion as the customary units.

Virtually all authorities recommend that the system of units be taught independently of the customary units, that is, no conversion between them. This is a sound position. We subscribe to it. However, even as adults, children in school today are likely to have to make some conversion between systems. We recommend that a limited, very limited, number of such equivalencies be learned by children. We suggest these: 1 in. = 2.54 cm , 1 mi. = 1.6 km , 1 oz. = 28 g , 1 A. = 0.4 ha , 1 qt. (liquid)=0.95ℓ. As was mentioned in Unit IX, rate paris make conversions rather easy, assuming that a person has some mastery of both systems.

$$12 \text{ in.} = \underline{\quad\quad}\text{cm} \quad \frac{\text{in}}{\text{cm}} \quad \frac{1}{2.54} = \frac{12}{\boxed{}} \boxed{} = 12 \times 2.54 = 30.48$$

$$656 \text{ g} = \underline{\quad\quad}\text{oz} \quad \frac{\text{oz}}{\text{g}} \quad \frac{1}{28} = \frac{\boxed{}}{655} \quad 28 \times \boxed{} = 665 \boxed{} = 23.4$$

"About" is a word that teachers should use often when referring to measures. "Exactly" should never be used with reference to measures, at least in practical terms like those arising from school and everyday life. All measures an approximations. "About" gives emphasis to that fact.

C. Review
1. Determine whether you have accomplished the objectives stated on page 284 of H & T.
2. Prepare any questions you have.

D. Assignment
1. Respond to exercises 1, 2, 4, 6, 8, 10, 12, 13, 15 (any one), and 16 on pages 312-314 in H & T.